Subject for this video:

The Closed Interval Method

Reading:

- **General:** Section 4.5 Absolute Maxima and Minima
- **More Specifically:** pages 298 – 299, Example 1

Homework:

H62: The Closed Interval Method (4.5#26,67)

Recall the definition of Critical Numbers from Section 4.1 (introduced in the Video for H55)

Definition of *Critical Number for* $f(x)$

Words: *critical number for* $f(x)$

Meaning: a number $x = c$ that satisfies these two requirements:

- The number $x = c$ is a partition number for $f'(x)$.
- The number $x = c$ is in the domain of $f(x)$.

That is,

- $f'(c) = 0$ or $f'(c)$ does not exist
- $f(c)$ exists

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Recall the definition Absolute Extrema from Section 4.5 (introduced in the Video for H61)

DEFINITION Absolute Maxima and Minima

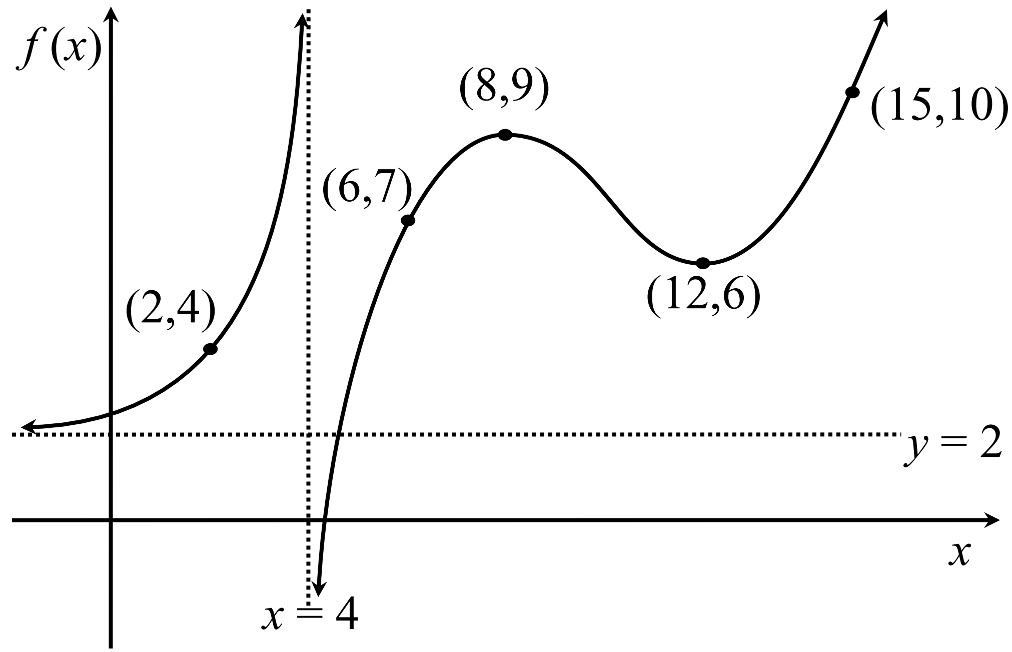
If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute maximum** of f. If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute minimum** of f. An absolute maximum or absolute minimum is called an **absolute extremum.**

And recall this example from the Video for H61

[Example 1]
(similar to 4.5#9,11,15,17,18)

The graph of a function $f(x)$ is shown.

Fill in the table below.



Interval	Local Maxes in that interval	Local Mins in that interval	Absolute Max in that interval	Absolute Min in that interval
$[6,15]$	$f(8) = 9$	$f(12) = 6$	$f(15) = 10$	$f(12) = 6$
$(6,15)$	$f(8) = 9$	$f(12) = 6$	none	$f(12) = 6$
$(8,15)$	none	$f(12) = 6$	none	$f(12) = 6$
$[12,15]$	none	none	$f(15) = 10$	$f(12) = 6$
$(-\infty, 4)$	none	none	none	none
$(4, \infty)$	$f(8) = 9$	$f(12) = 6$	none	none

Notice that for some of the intervals, $f(x)$ some of the types of extrema do not occur.

But there is one important situation where some extrema are *guaranteed* to occur.

THEOREM 1 Extreme Value Theorem
A function f that is continuous on a closed interval $[a, b]$ has both an absolute maximum and an absolute minimum on that interval.

Remember that *local extrema* can only occur at *critical numbers* of $f(x)$. (*not at endpoints*.)

But notice that *absolute extrema can occur* at *endpoints*.

THEOREM 2 Locating Absolute Extrema
Absolute extrema (if they exist) must occur at critical numbers or at endpoints.

Theorems 1 and 2 are the basis for the following *procedure* for finding the *absolute extrema* on a *closed interval* for a function that is *continuous* on that interval.

The *Closed Interval Method*

PROCEDURE Finding Absolute Extrema on a Closed Interval

Step 1 Check to make certain that f is continuous over $[a, b]$.
Step 2 Find the critical numbers in the interval (a, b).
Step 3 Evaluate f at the endpoints a and b and at the critical numbers found in step 2.
Step 4 The absolute maximum of f on $[a, b]$ is the largest value found in step 3.
Step 5 The absolute minimum of f on $[a, b]$ is the smallest value found in step 3.

Evaluate f at c means find $f(c)$

[Example 1](similar to 4.5#26,67)

(a) Find the absolute extrema of $f(x) = x^3 - 3x^2 - 9x + 13$ on the interval $[-2,5]$.

Step 1 Domain $[-2,5]$ is a closed interval ✓

$f(x)$ is continuous on the domain. (polynomial) ✓

Step 2 Find Critical Numbers of $f(x)$

$$f'(x) = \frac{d}{dx}\left(x^3 - 3x^2 - 9x + 13\right) = 3x^2 - 6x - 9 =$$

$$= 3(x^2 - 2x - 3) = 3(x+1)(x-3)$$

$f'(x) = 0$ when $x = -1$ and when $x = -3$

There are no x values that cause f' to not exist because $f'(x)$ is a polynomial.

So the partition numbers for $f'(x)$ are $x=-1, x=3$,

These are also the critical numbers for $f(x)$ because $f(x)$ always exists. (polynomial.

Critical numbers for $f(x)$: $x = -1$, $x = 3$

Step 3

important x values	Corresponding y value $f(x)=x^3-3x^2-9x+13$
$x=-2$ (endpoint)	$f(-2)=(-2)^3-3(-2)^2-9(-2)+13=\dots=11$
$x=-1$ (critical)	$f(-1)=(-1)^3-3(-1)^2-9(-1)+13=\dots=18$
$x=3$ (critical)	$f(3)=(3)^3-3(3)^2-9(3)+13=\dots=-14$
$x=5$ (endpoint)	$f(5)=(5)^3-3(5)^2-9(5)+13=\dots=18$

Step 4+5

The absolute max is $y=18$. It occurs at $x=-1$ and $x=5$

The absolute min is $y=-14$. It occurs at $x=3$.

Observations:

- Use the formula for $f(x)$ to compute the y values. (Don't use the formula for $f'(x)$!)
- Absolute extrema can occur at more than one x value.
- Absolute extrema can occur at critical numbers and at endpoints.
- The *Closed Interval Method* does not use a sign chart for $f'(x)$ to locate the extrema. The absolute extrema must occur at the important x values on the list. A sign chart for $f'(x)$ is not needed, so don't waste time making one

(b) Find the absolute extrema of $f(x) = x^3 - 3x^2 - 9x + 13$ on the interval $[-2,2]$.

Solution: We already know that the critical numbers for $f(x)$ are

$x = -1$, ~~$x = 3$~~ not in our interval

List of important X values	Corresponding y value $f(x) = x^3 - 3x^2 - 9x + 13$
$x = -2$ (endpoint)	$f(-2) = 11$ (from part (a))
$x = -1$ (critical)	$f(-1) = 18$ (from part (a))
$x = 2$ (endpoint)	$f(2) = (2)^3 - 3(2)^2 - 9(2) + 13 = \ldots = -9$

The absolute max is $y = 18$. It occurs at $x = -1$

The absolute min is $y = -9$. It occurs at $x = 2$

Observations:

- Absolute extrema depend on the choice of interval.
- The list of important x values does not include critical numbers outside the interval.

[Example 2] (similar to 4.5#26,67)(Function from Video for H60) $f(x) = -x^4 + 6x^2 + 27$.

Find the absolute extrema of $f(x) = -x^4 + 6x^2 + 27$ on the interval $[-2,3]$.

Step 1 f is continuous (it is a polynomial) and the domain is a closed interval

Step 2 Critical numbers for f(x) will be the x values that cause $f'(x) = 0$,

$$f'(x) = \frac{d}{dx}\left(-x^4 + 6x^2 + 27\right) = -4x^3 + 12x =$$

$$= -4x\left(x^2 - 3\right) = -4x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$$

$f'(x) = 0$ when $x = 0$, $x = -\sqrt{3}$, $x = \sqrt{3}$

These are the partition numbers for $f'(x)$ and the critical numbers for $f(x)$

Step 3

Important X values	Corresponding y values $f(x) = -x^4 + 6x^2 + 27$
$x = -2$ (endpoint)	$f(-2) = -(-2)^4 + 6(-2)^2 + 27 = \dots = 35$
$x = -\sqrt{3}$ (critical)	$f(-\sqrt{3}) = -(-\sqrt{3})^4 + 6(-\sqrt{3})^2 + 27 = \dots = 36$
$x = 0$ (critical)	$f(0) = -(0)^4 + 6(0)^2 + 27 = 27$
$x = \sqrt{3}$ (critical)	$f(\sqrt{3}) = -(\sqrt{3})^4 + 6(\sqrt{3})^2 + 27 = \dots = 36$
$x = 3$ (endpoint)	$f(3) = -(3)^4 + 6(3)^2 + 27 = \dots = 0$

Step 4 & 5

The absolute max is $y = 36$, It occurs at $x = -\sqrt{3}$ and $x = \sqrt{3}$

The absolute min is $y = 0$, It occurs at $x = 3$

Observations:

- Absolute extrema can occur at x values that are not integers

Consider Ann's solution to the question posed in [Example 2].

Find the absolute extrema of $f(x) = -x^4 + 6x^2 + 27$ on the interval $[-2,3]$.

Ann made a list of y values.

x	$f(x) = -x^4 + 6x^2 + 27$
-2	$f(-2) = -(-2)^4 + 6(-2)^2 + 27 = -(16) + 6(4) + 27 = -16 + 24 + 27 = 35$
-1	$f(-1) = -(-1)^4 + 6(-1)^2 + 27 = -(1) + 6(1) + 27 = -1 + 6 + 27 = 32$
0	$f(0) = -(0)^4 + 6(0)^2 + 27 = 27$
1	$f(1) = -(1)^4 + 6(1)^2 + 27 = -(1) + 6(1) + 27 = -1 + 6 + 27 = 32$
2	$f(2) = -(2)^4 + 6(2)^2 + 27 = -(16) + 6(4) + 27 = -16 + 24 + 27 = 35$
3	$f(3) = -(3)^4 + 6(3)^2 + 27 = -(81) + 6(9) + 27 = -81 + 54 + 27 = 0$

Based on this list, Ann makes the following conclusion

- The *absolute max* is $y = 35$ and it occurs at $x = -2$ and $x = 2$.
- The *absolute min* is $y = 0$ and it occurs at $x = 3$.

Ann's Method is Invalid!

Observations:

- Ann got the wrong *absolute max*, because she only considered integer x values.
- Ann happened to get the right value for the *absolute min*, but her method does not actually *prove* that $y = 0$ is the absolute min, because she did not investigate the critical numbers.
- In other words, Ann's answer about the *absolute min* (an answer that happens to be *correct*) is no more valid than her answer about the *absolute max* (an answer that happens to be *incorrect*). The method that Ann used is an *invalid method*, even if it may happen to give the correct answer in some situations. Even though it involved a lot of work, it is *not a valid method* for finding absolute extrema.

Consider Bob's solution to the question posed in [Example 2].

Find the absolute extrema of $f(x) = -x^4 + 6x^2 + 27$ on the interval $[-2,3]$.

Bob found the partition numbers for $f'(x)$

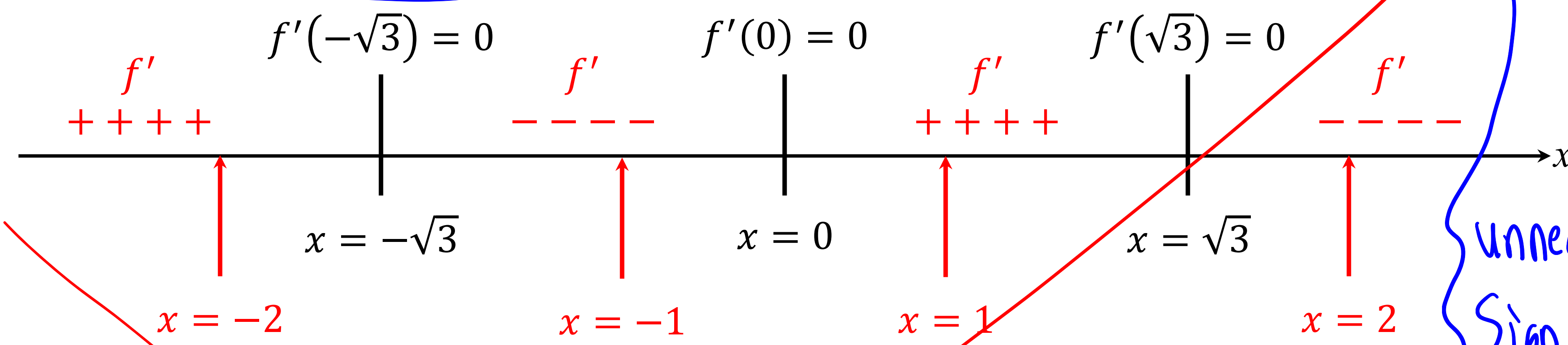
$$f'(x) = \frac{d}{dx}(-x^4 + 6x^2 + 27) = -4x^3 + 12x = -4x(x^2 - 3) - 4x(x + \sqrt{3})(x - \sqrt{3})$$

Since $f'(x)$ is a polynomial, there are no x values that cause $f'(x)$ to not exist.

Observe that $f'(x) = 0$ at $x = -\sqrt{3}$ and $x = 0$ and $x = \sqrt{3}$.

So the partition numbers for $f'(x)$ are $x = -\sqrt{3}$ and $x = 0$ and $x = \sqrt{3}$.

Then Bob made a sign chart for $f'(x)$



$f'(-2) = -4(-2)((-2)^2 - 3) = -4(-2)(4-3) = -4(-2)(1) = pos$

$f'(-1) = -4(-1)((-1)^2 - 3) = -4(-1)(1-3) = -4(-1)(-2) = neg$

$f'(1) = -4(1)((1)^2 - 3) = -4(1)(1-3) = -4(1)(-2) = pos$

$f'(2) = -4(2)((2)^2 - 3) = -4(2)(4-3) = -4(2)(1) = neg$

unnecessary sign chart

Based on this sign chart, Bob said that there is a max at $x = -\sqrt{3}$ and $x = \sqrt{3}$ and a min at $x = 0$.

Bob computed the y values at those x values.

$$f(-\sqrt{3}) = -(-\sqrt{3})^4 + 6(-\sqrt{3})^2 + 27 = -(9) + 6(3) + 27 = 36 \ (abs\ max)$$ correct

$$f(0) = -(0)^4 + 6(0)^2 + 27 = 27 \ (abs\ min)$$ incorrect

$$f(\sqrt{3}) = -(\sqrt{3})^4 + 6(\sqrt{3})^2 + 27 = -(9) + 6(3) + 27 = 36 \ (abs\ max)$$ correct

invalid method

Observations:

- Bob got the wrong *absolute min*, because he did not investigate the endpoints.
- Bob happened to get the right value for the *absolute max*, but his method does not actually *prove* that $y = 36$ is the absolute max, because he did not investigate the endpoints.
- In other words, Bob's answer about the *absolute max* (an answer that happens to be *correct*) is no more valid than his answer about the *absolute min* (an answer that happens to be *incorrect*). The method that Bob used is an *invalid method*, even if it may happen to give the correct answer in some situations. Even though it involved a lot of work, it is *not a valid method* for finding absolute extrema.
- All of the work that Bob did in making the sign chart for $f'(x)$ is *unnecessary*. All that is needed is to find the important x values
    - The endpoints of the interval
    - the critical numbers for $f(x)$ that are in the interval.

  and then compute the y values at those important x values.