Subject for this video:

Finding Absolute Extrema on General Intervals; the 2nd Derivative Test

Reading:

- **General:** Section 4.5 Absolute Maxima and Minima
- **More Specifically:** pages 299 – 302, Examples 2, 3

Homework:

H63: Finding Absolute Extrema on General Intervals; the 2nd Derivative Test

(4.5#35,43,49,51,53*,73,79)

Recall the definition of Critical Numbers from Section 4.1 (introduced in the Video for H55)

Definition of *Critical Number for* $\boldsymbol{f(x)}$

Words: *critical number for* $f(x)$

Meaning: a number $x = c$ that satisfies these two requirements:

- The number $x = c$ is a partition number for $f'(x)$.
- The number $x = c$ is in the domain of $f(x)$.

That is,

- $f'(c) = 0$ or $f'(c)$ does not exist
- $f(c)$ exists

.

Recall the definition Absolute Extrema from Section 4.5 (introduced in the Video for H61)

DEFINITION Absolute Maxima and Minima

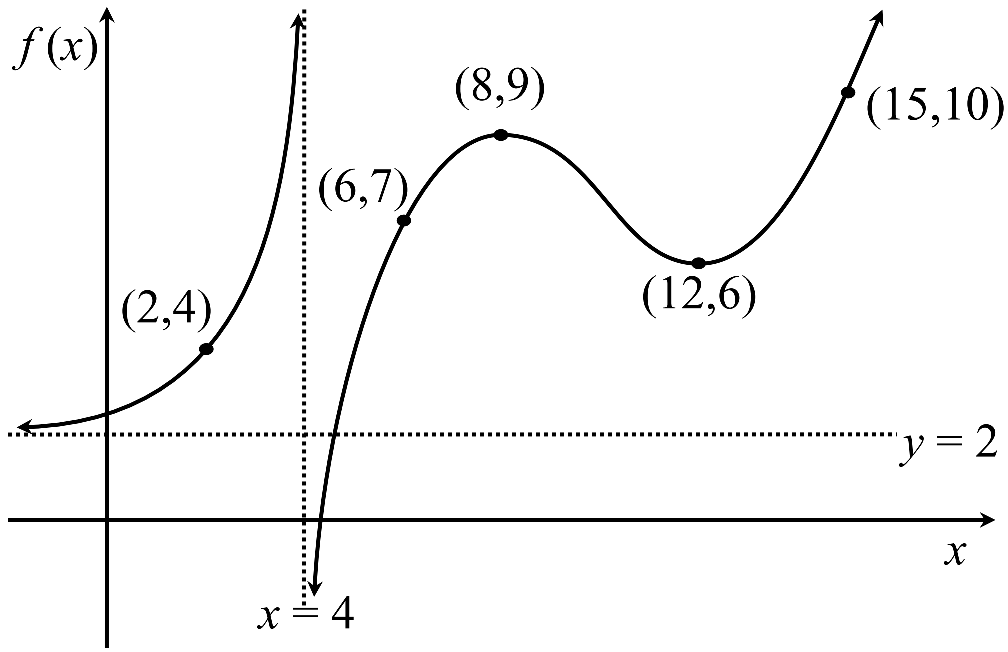
If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute maximum** of f. If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute minimum** of f. An absolute maximum or absolute minimum is called an **absolute extremum.**

And recall this example from the Video for H61, illustrating that absolute extrema do not always occur.

[Example 1]
(similar to 4.5#9,11,15,17,18)

The graph of a function $f(x)$ is shown.

Fill in the table below.



Interval	Local Maxes in that interval	Local Mins in that interval	Absolute Max in that interval	Absolute Min in that interval
$[6,15]$	$f(8) = 9$	$f(12) = 6$	$f(15) = 10$	$f(12) = 6$
$(6,15)$	$f(8) = 9$	$f(12) = 6$	none	$f(12) = 6$
$(8,15)$	none	$f(12) = 6$	none	$f(12) = 6$
$[12,15]$	none	none	$f(15) = 10$	$f(12) = 6$
$(-\infty, 4)$	none	none	none	none
$(4, \infty)$	$f(8) = 9$	$f(12) = 6$	none	none

Recall the Closed Interval Method (introduced the Video for H62)

There is one important situation where *both absolute max and absolute min are guaranteed*.

> **THEOREM 1 Extreme Value Theorem**
>
> A function f that is continuous on a closed interval $[a, b]$ has both an absolute maximum and an absolute minimum on that interval.

And there is a theorem that tells us where Absolute Extrema *have to occur*.

> **THEOREM 2 Locating Absolute Extrema**
>
> Absolute extrema (if they exist) must occur at critical numbers or at endpoints.

Theorems 1 and 2 are the basis for the following *procedure* (the ***Closed Interval Method***) for finding the *absolute extrema* on a *closed interval* for a function that is *continuous* on that interval.

> **PROCEDURE Finding Absolute Extrema on a Closed Interval**
>
> Step 1 Check to make certain that f is continuous over $[a, b]$.
> Step 2 Find the critical numbers in the interval (a, b).
> Step 3 Evaluate f at the endpoints a and b and at the critical numbers found in step 2.
> Step 4 The absolute maximum of f on $[a, b]$ is the largest value found in step 3.
> Step 5 The absolute minimum of f on $[a, b]$ is the smallest value found in step 3.

But what about the situation where the domain of the function is *not* a closed interval? How does one determine the absolute extrema that *do* occur?

As we will see in this video, that question is answered in different ways for different functions.

For some familiar function types, the approach can be to

- First, consider the end behavior to determine which kinds of absolute extrema will occur.
- Then, find the locations of those extrema.

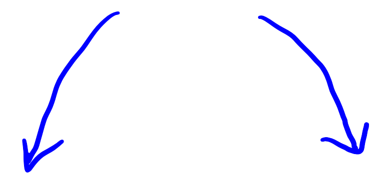
[Example 1] (Similar to 4.5#43,49) For the function $f(x) = -3x^4 - 4x^3 + 36x^2 + 5$

(A) Find all absolute extrema of $f(x)$ on the interval $(-\infty, \infty)$

Notice: Domain is not a closed interval, so we are not guaranteed any extrema.

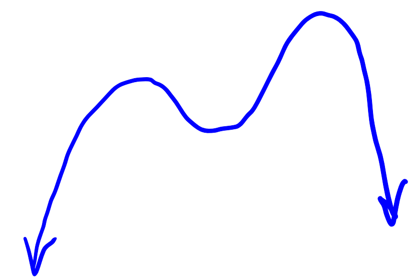
The end behavior will be

even degree polynomial
negative leading coefficient } so graph will go down on both ends.



f is continuous, so graph will have no breaks or jumps or vertical asymptotes.

Graph could have up to 3 turning points



Conclude there will not be an absolute min but there will be an absolute max.

Now we must find the absolute max.

Theorem 2 tells us that it will occur at a critical number for $f(x)$

So find critical numbers for $f(x) = -3x^4 - 4x^3 + 36x^2 + 5$

$$f'(x) = \frac{d}{dx}\left(-3x^4 - 4x^3 + 36x^2 + 5\right) = -3(4x^3) - 4(3x^2) + 36(2x) + 0$$

$$= -12x^3 - 12x^2 + 72x = -12x(x^2 + x - 6)$$

$$= -12x(x+3)(x-2)$$

$f'(x) = 0$ when $x = 0,\ x = -3,\ x = 2$

These are the only partition numbers for $f'(x)$ and are the critical numbers for $f(x)$.

Compare y values for $f(x) = -3x^4 - 4x^3 + 36x^2 + 5$

$f(-3) = -3(-3)^4 - 4(-3)^3 + 36(-2)^2 + 5 = \cdots = 194$

$f(0) = -3(0)^4 - 4(0)^3 + 36(0)^2 + 5 = 5$

$f(2) = -3(2)^4 - 4(2)^3 + 36(2)^2 + 5 = \cdots = 69$

The absolute max on the interval $(-\infty, \infty)$ is $y = 194$

It occurs at $x = -3$

There is no absolute min on the interval $(-\infty, \infty)$

(B) Find all absolute extrema of $f(x)$ on the interval $[-2, \infty)$

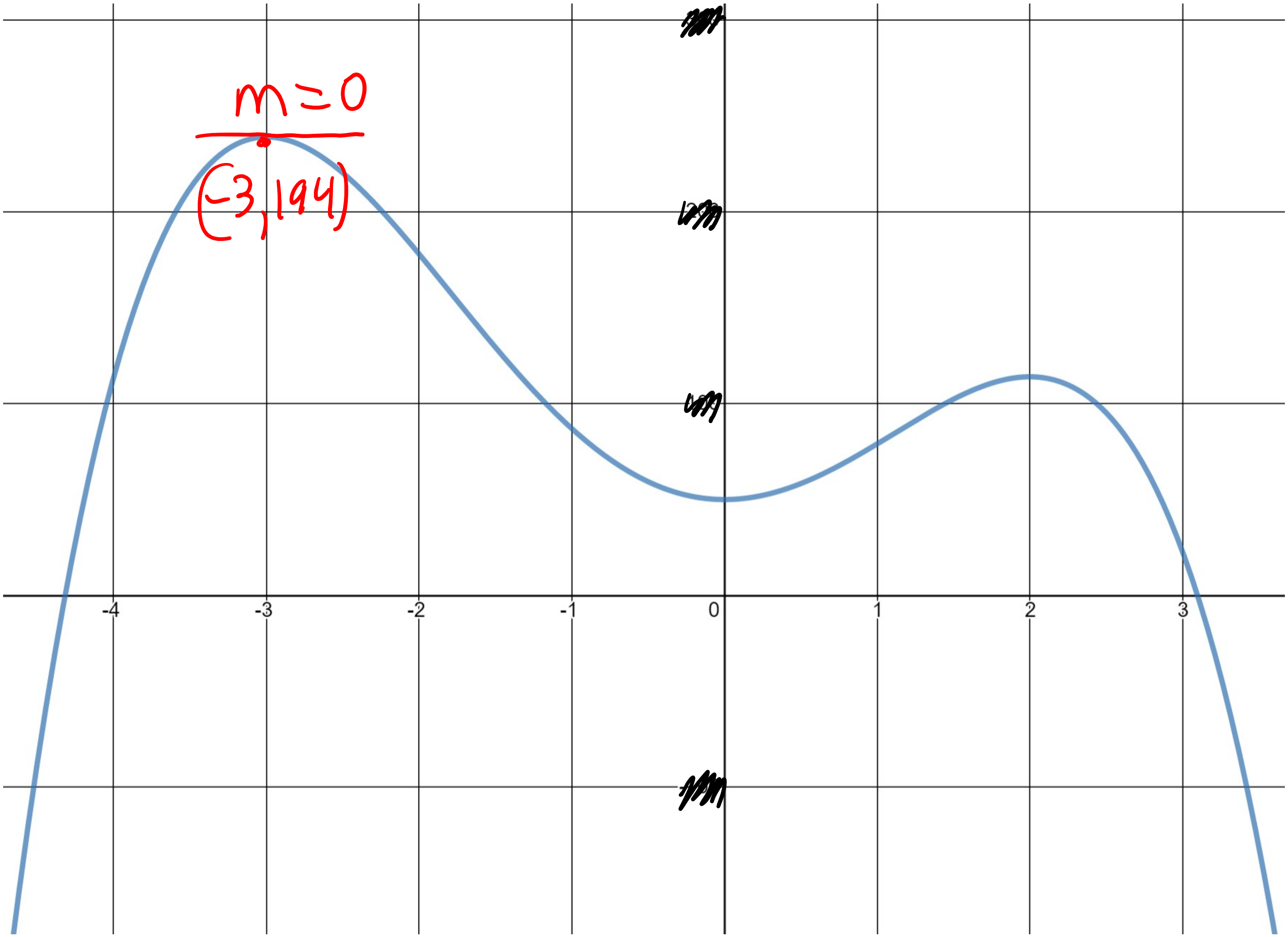
Solution The left end of our graph got cut off, but the graph still goes down on the right. So there will not be an absolute min. But there will still be an absolute max.

Theorem 2 tells us that the abs max must occur at either an endpoint or a critical number in the interval.

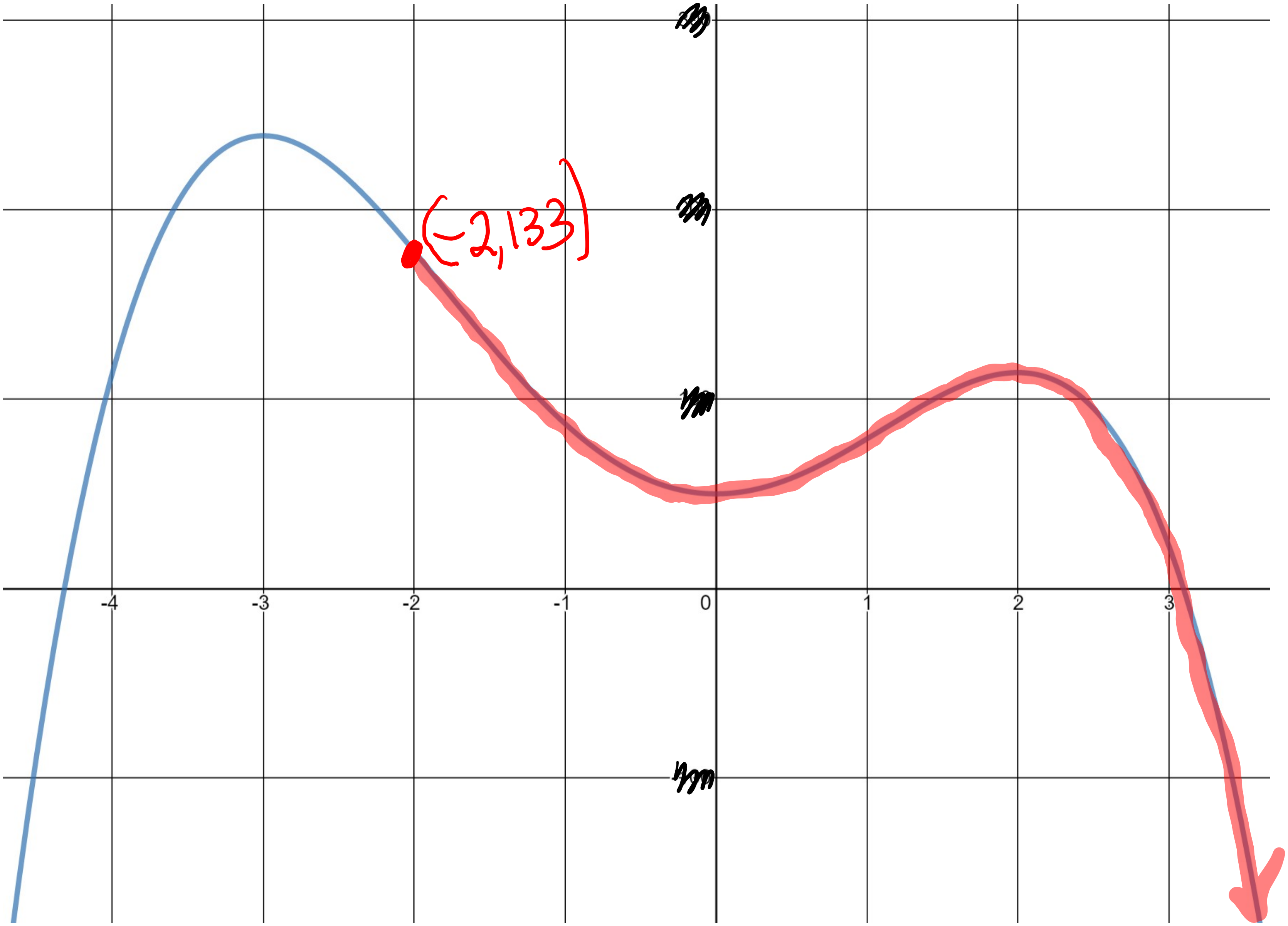
Important X values	Corresponding y value
$x = -2$ (endpoint)	$f(-2) = -3(-2)^4 - 4(-2)^3 + 36(-2)^2 + 5 = \dots = 133$
$x = 0$ (critical)	$f(0) = 5$ from part (a)
$x = 2$ (critical)	$f(2) = 69$ from part (a)

Conclude that the absolute max is $y = 133$. It occurs at $x = -2$.
There is no absolute min

(C) Illustrate your results from **(A)** on the given graph of $f(x)$.



(D) Illustrate your results from **(B)** on the given graph of $f(x)$.



The example that we just completed was a familiar function type (a polynomial), and so we were able to determine which kinds of absolute extrema would occur before doing any calculations, by simply reasoning in terms of the *graph shape*.

Of course, the fact that we did not have to do any calculations to determine which kinds of extrema would occur does not mean that that determination is *easy*. One must have the wisdom to know that the approach should be to reason in terms of the graph shape.

What about cases where the function is *not* a familiar type, so one *cannot* determine which types of absolute extrema will occur by simply reasoning in terms of the *graph shape*?

[Example 2] (Similar to 4.5#35,51,53) For the function $f(x) = 55 - 4x - \frac{250}{x^2}$

(A) Find all absolute extrema of $f(x)$ on the interval $(0, \infty)$

Observe. Not a familiar function type
Not a closed interval,

So we are not guaranteed there will be any extrema, and we don't know the general shape of the graph.

Theorem 2 says that if there are any extrema, they must occur at critical numbers.

So, find critical numbers.

Critical numbers for $f(x) = 55 - 4x - \frac{250}{x^2}$

Convert $f(x)$ to power function form

$$f(x) = \underbrace{55 - 4x - \frac{250}{x^2}}_{\text{positive exponent form}} = \underbrace{55 - 4x - 250x^{-2}}_{\text{power function form}}$$

Now find $f'(x)$

$$f'(x) = \frac{d}{dx}\left(55 - 4x - 250x^{-2}\right) = 0 - 4(1) - 250\left(-2x^{-2-1}\right)$$

$$= \underbrace{-4 + 500x^{-3}}_{\text{power function form}} = \underbrace{-4 + \frac{500}{x^3}}_{\text{positive exponent form}}$$

Partition numbers for $f'(x) = -4 + \frac{500}{x^3}$

Observe $f'(0) = -4 + \frac{500}{(0)^3}$ does not exist,

So $x = 0$ is a partition number for $f'(x)$

Are there any x values that cause $f'(x) = 0$?

$$f'(x) = -4 + \frac{500}{x^3} = 0$$

$$\frac{500}{x^3} = 4$$

$$x^3 = \frac{500}{4} = 125$$

$$x = 5$$

So $x = 5$ is a partition number for $f'(x)$ because $f'(5) = 0$

Partition numbers for $f'(x)$ are

$x=0$ because $f'(0)$ DNE

$x=5$ because $f'(5)=0$

Are either of these qualified to be called critical numbers for $f(x)$? See if $f(x)$ exists,

$$f(x) = 55 - 4x - \frac{250}{x^2}$$

$$f(0) = 55 - 4(0) - \frac{250}{0^2} \quad \text{DNE}$$

$$f(5) = 55 - 4(5) - \frac{250}{(5)^2} = 55 - 20 - \frac{250}{25}$$

$$= 55 - 20 - 10$$

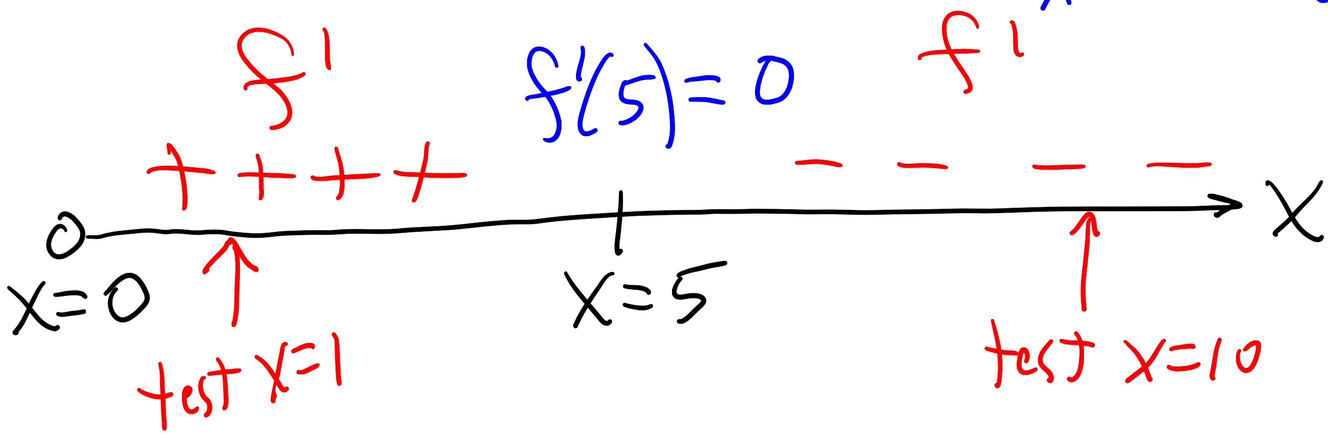
$$= 25$$ this exists.

So $x=5$ is the only critical number for $f(x)$

If there is an abs max or min for $f(x)$, it must occur at the critical number $x=5$;

Strategy now: • Make sign chart for $f'(x)$
• use it to figure out increasing & decreasing behavior of $f(x)$

Sign chart for $f'(x) = -4 + \frac{500}{x^3}$ on interval $(0,\infty)$



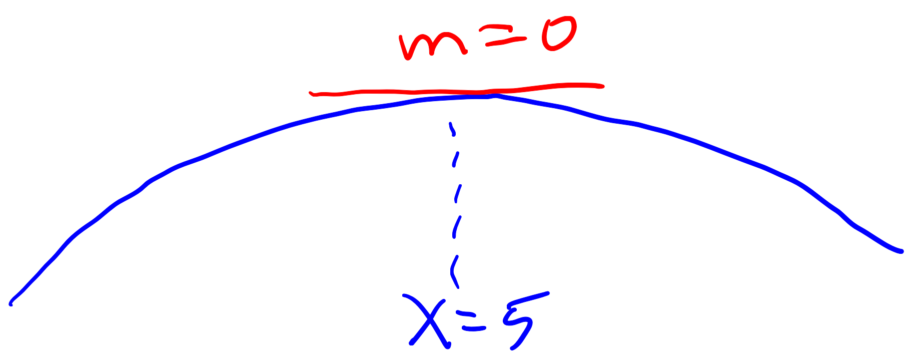
$f'(1) = -4 + \frac{500}{(1)^3} = -4 + 500 = \text{pos}$

$f'(10) = -4 + \frac{500}{(10)^3} = -4 + \frac{500}{1000} = -4 + \frac{1}{2} = \text{neg}$

So $f(x)$ is increasing on $(0,5)$

has horiz tangent at $x=5$

and then decreasing on $(5,\infty)$

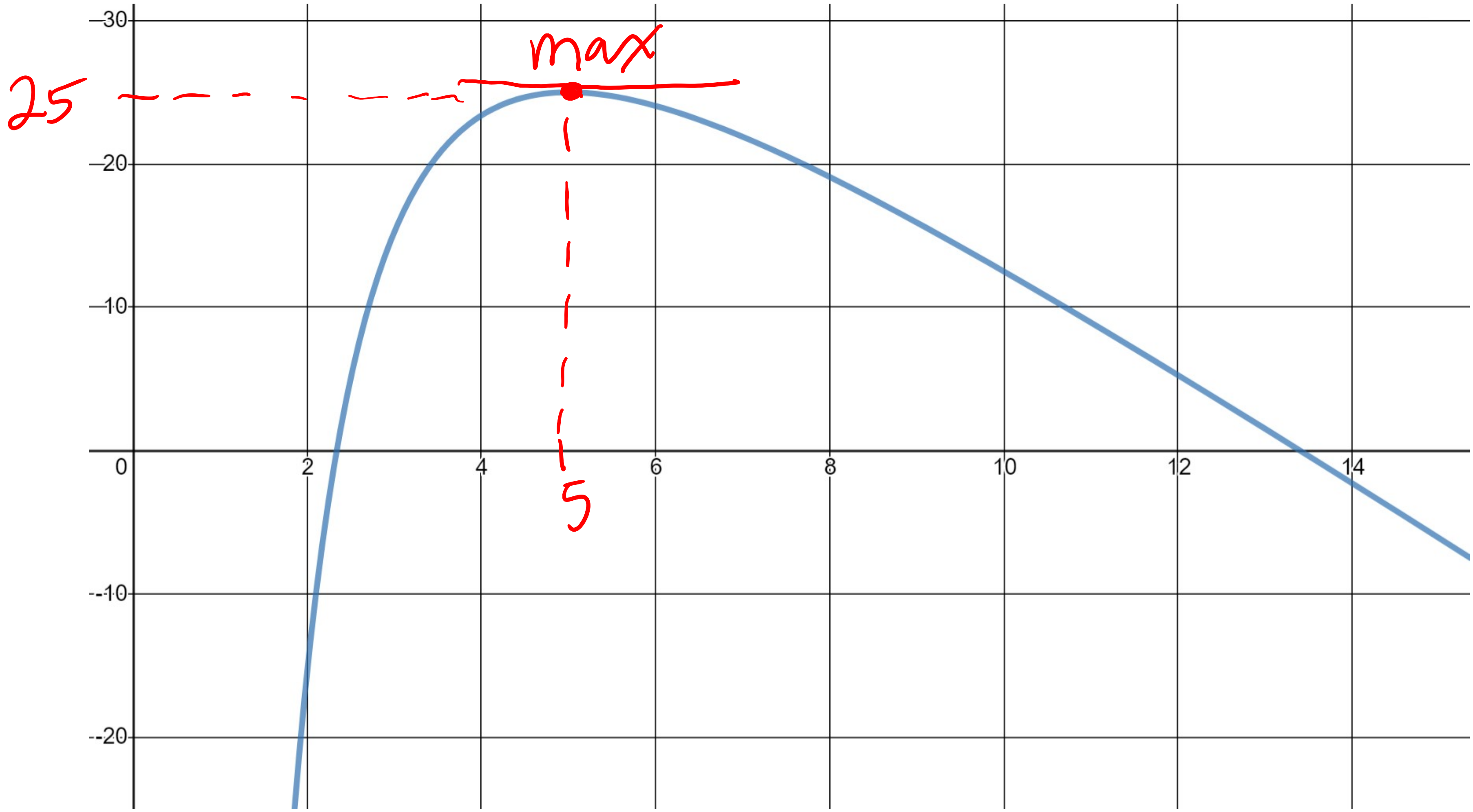


Conclude there will be an absolute max at $x=5$.

So on the interval $(0,\infty)$, $f(x)$ has an absolute max of $y=25$. It occurs at $x=5$

$f(x)$ does not have an absolute min

(B) Illustrate on the given graph of $f(x)$.



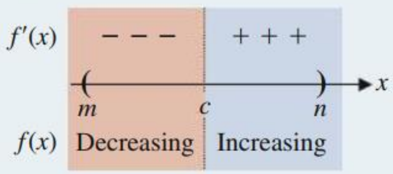
The Second-Derivative Test for Local Extrema

In Section 4.1, we learned how to determine the location of local maxes and mins for a function $f(x)$ by studying the sign behavior of $f'(x)$. We used the *First-Derivative Test for Local Extrema* to reach our conclusions.

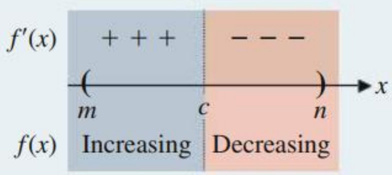
PROCEDURE First-Derivative Test for Local Extrema

Let c be a critical number of f [$f(c)$ is defined and either $f'(c) = 0$ or $f'(c)$ is not defined]. Construct a sign chart for $f'(x)$ close to and on either side of c.

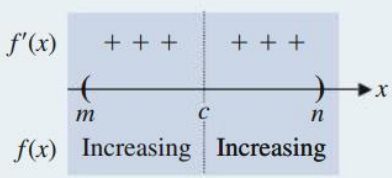
Sign Chart | ***f*(*c*)**



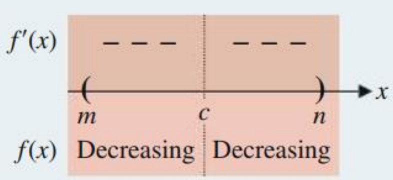
$f(c)$ is a local minimum.
If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a local minimum.



$f(c)$ is a local maximum.
If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a local maximum.

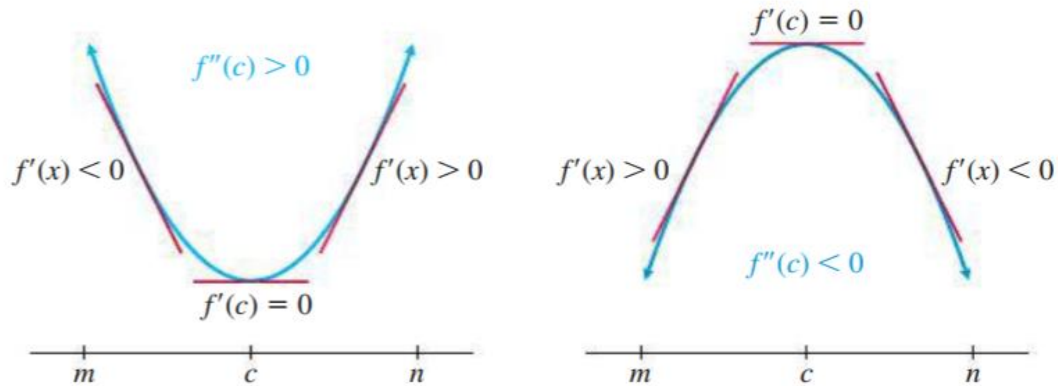


$f(c)$ is not a local extremum.
If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.



$f(c)$ is not a local extremum.
If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.

There is another useful tool for determining the location of local maxes and mins for a function $f(x)$, one that involves studying the sign behavior of $f'(x)$ and $f''(x)$. It is called the *Second Derivative Test for Local Extrema*.



(A) $f'(c) = 0$ and $f''(c) > 0$ implies $f(c)$ is a local minimum

(B) $f'(c) = 0$ and $f''(c) < 0$ implies $f(c)$ is a local maximum

Figure 4 Second derivative and local extrema

RESULT Second-Derivative Test for Local Extrema

Let c be a critical number of $f(x)$ such that $f'(c) = 0$. If the second derivative $f''(c) > 0$, then $f(c)$ is a local minimum. If $f''(c) < 0$, then $f(c)$ is a local maximum.

$f'(c)$	$f''(c)$	Graph of f is:	$f(c)$	Example
0	+	Concave upward	Local minimum	
0	−	Concave downward	Local maximum	
0	0	?	Test does not apply	

Observe that the *Second-Derivative Test* does not apply to situations where both $f'(c) = 0$ and $f''(c) = 0$. It is useful to think about why it would not apply.

Consider the functions of $f(x) = x^4 + 5$ and $g(x) = -x^4 + 5$ and $\cancel{h(x) = x^3 + 5}$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$g'(x) = -4x^3$

$g''(x) = 12x^2$

$h(x) = x^3 + 5$

$h'(x) = 3x^2$

$h''(x) = 6x$

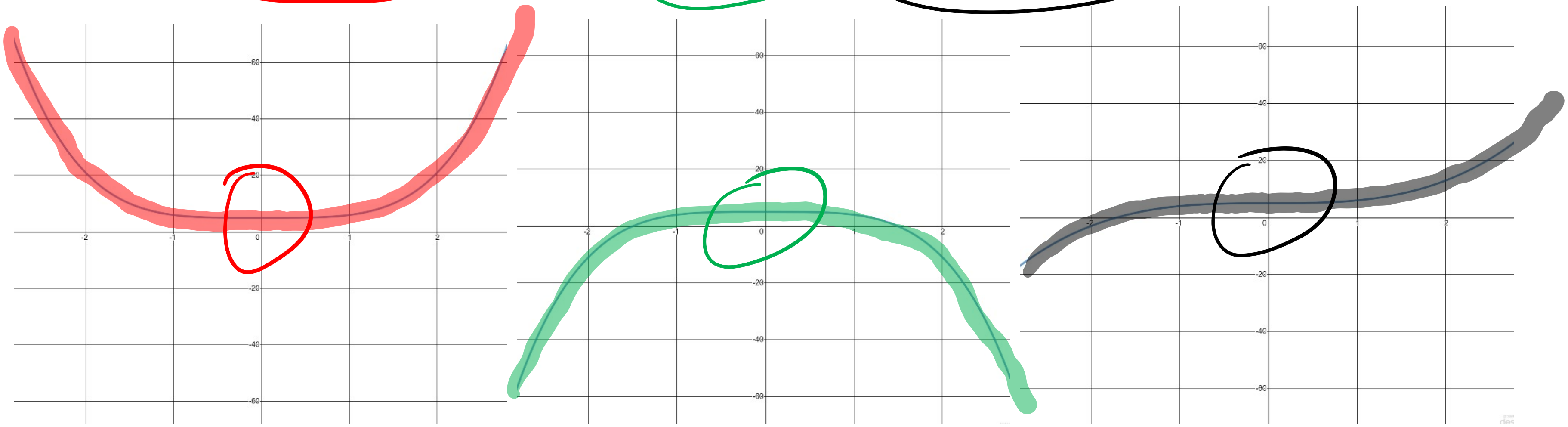
Observe that

- $f'(0) = 0$ and $f''(0) = 0$.
- $g'(0) = 0$ and $g''(0) = 0$.
- $h'(0) = 0$ and $h''(0) = 0$.

The *Second-Derivative Test for Local Extrema* does not apply to these functions at $x = 0$.

Now consider the graphs of the functions, shown on the next page.

Graphs of $f(x) = x^4 + 5$ and $g(x) = -x^4 + 5$ and $h(x) + x^3 + 5$



Observe that

- $f(x)$ has a local min at $x = 0$.
- $g(x)$ has a local max at $x = 0$.
- $h(x)$ does not have a max or min at $x = 0$.

So we see that it is possible for a function to have 1st and 2nd derivatives both equal to zero at some $x = c$, and for that function to have a local min, or a local max, or neither at $x = c$.

That's why the 2nd-*Derivative Test* does not apply in such a case. Note that in such a case, one would use the 1st-*Derivative Test* to determine if there was a local max or min at $x = c$.

[Example 3] (Similar to 4.5#73,79) Suppose that it is known that a function $f(x)$ is continuous on an interval containing $x = 7$ and that $f'(7) = 0$ and $f''(7) < 0$.

Describe the behavior of the graph of $f(x)$ at the point $(x, y) = \big(7, f(7)\big)$.

$f(x)$ has a local max at $x=7$,

The value of the max is

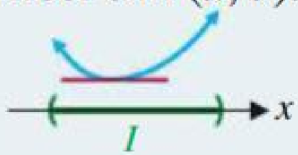
the y value $y = f(7)$

The Second-Derivative Test for Absolute Extrema

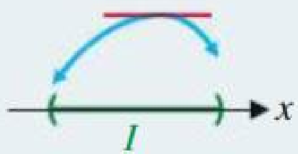
THEOREM 3 Second-Derivative Test for Absolute Extrema on an Interval

Let f be continuous on an interval I from a to b with only one critical number c in (a, b).

If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on I.



If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on I.



[Example 2](revisited) (Similar to 4.5#35,51,53) studying the function $f(x) = 55 - 4x - \frac{250}{x^2}$

(C) In part **(A)** of this example, it was found that $f'(5) = 0$. Then a sign chart for $f'(x)$ was made. That sign chart was used to determine that on the interval $(0, \infty)$, the function $f(x)$ has an absolute max at $x = 5$ and no absolute min.

Redo that investigation using the *Second-Derivative Test for Absolute Extrema*. That is, having determined that $f'(5) = 0$, don't proceed to make a sign chart for $f'(x)$. Instead, use the *Second-Derivative Test* to determine the absolute extrema.

We found $f'(x) = \underbrace{-4 + 500x^{-3}}_{\text{power function form}} = \underbrace{-4 + \frac{500}{x^3}}_{\text{positive exponent form}}$

find $f''(x) = \frac{d}{dx}\left(-4 + 500x^{-3}\right) = 0 + 500(-3)x^{-3-1} =$

$= -1500x^{-4} = -\frac{1500}{x^4}$

find sign of $f''(5)$

$f''(5) = \frac{-1500}{(5)^4}$ = negative

Since $f'(5)=0$ and $f''(5)=$ negative

(and since $x=5$ is the <u>only</u> critical number on the interval $(0,\infty)$)

we can conclude (by the Second-Derivative Test for Absolute Extrema) that $f(x)$ has an <u>absolute</u> <u>max</u> at $x=5$.