Subject for this video:

Single Variable Optimization Problems about Maximizing Revenue and Profit

Reading: 4.6 Optimization

- **General:** Section ~~4.5 Absolute Maxima and Minima~~
- **More Specifically:** pages 307 – 311 Examples 3,4,6,7

Homework: H64: Single Variable Optimization Problems about Maximizing Revenue and Profit (4.6#19,25,27)

Recall the definition of Critical Numbers from Section 4.1 (introduced in the Video for H55)

Definition of *Critical Number for $f(x)$*

Words: *critical number for $f(x)$*

Meaning: a number $x = c$ that satisfies these two requirements:

- The number $x = c$ is a partition number for $f'(x)$.
- The number $x = c$ is in the domain of $f(x)$.

That is,

- $f'(c) = 0$ or $f'(c)$ does not exist
- $f(c)$ exists

Recall the definition Absolute Extrema from Section 4.5 (introduced in the Video for H61)

DEFINITION Absolute Maxima and Minima

If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute maximum** of f. If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute minimum** of f. An absolute maximum or absolute minimum is called an **absolute extremum.**

And these Theorems about absolute extrema from Section 4.5 (introduced the Video for H62)

There is one important situation where *both absolute max and absolute min are guaranteed*.

THEOREM 1 Extreme Value Theorem

A function f that is continuous on a closed interval $[a, b]$ has both an absolute maximum and an absolute minimum on that interval.

And there is a theorem that tells us where Absolute Extrema *have to occur*.

THEOREM 2 Locating Absolute Extrema

Absolute extrema (if they exist) must occur at critical numbers or at endpoints.

Theorems 1 and 2 are the basis for the following *procedure* (the ***Closed Interval Method***) for finding the *absolute extrema* on a *closed interval* for a function that is *continuous* on that interval. This procedure was discussed in the Video for Homework H62.

PROCEDURE Finding Absolute Extrema on a Closed Interval

Step 1 Check to make certain that f is continuous over $[a, b]$.
Step 2 Find the critical numbers in the interval (a, b).
Step 3 Evaluate f at the endpoints a and b and at the critical numbers found in step 2.
Step 4 The absolute maximum of f on $[a, b]$ is the largest value found in step 3.
Step 5 The absolute minimum of f on $[a, b]$ is the smallest value found in step 3.

But what about the situation where the domain of the function is *not* a closed interval? How does one determine the absolute extrema that *do* occur? As we will saw in the video for Homework H63, that question is answered in different ways for different functions.

For some familiar function types, the approach can be to

- First, consider the end behavior to determine which kinds of absolute extrema will occur.
- Then, find the locations of those extrema in the following way:
    - Find the critical numbers of the function in the domain
    - Compute values of $f(x)$ at those critical numbers and at endpoints (if there are any)
    - Identify the absolute max or min values that you know will occur.

For functions that are not familiar function types, the approach is to

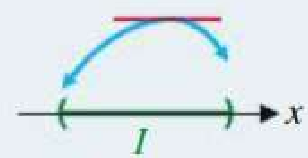
- Find the critical numbers of the function in the domain.
- Determine if any of those critical numbers is the location of an absolute extremum by either
  - studying the sign behavior of $f'(x)$ to determine increasing/decreasing behavior of $f(x)$
  - or using the Second Derivative Test for Absolute Extrema on an Interval

THEOREM 3 Second-Derivative Test for Absolute Extrema on an Interval

Let f be continuous on an interval I from a to b with only one critical number c in (a, b).

If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on I.

If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on I.



Optimization

In Section 4.6, we will study problems involving *Optimization*.

Optimization problems are simply Absolute Max/Min problems, but they may have complications

- They may be presented as word problems, about applications to real world situations.
- You may have to figure out a mathematical model.
- The initial mathematical model may involve more than one variable. If it does, then you will have to figure out a function of one variable.
- The domain might not be a closed interval

Homework 64 consists of

Single Variable Optimization Problems about Maximizing Revenue and Profit

This video is about those kinds of problems.

[Example 1] (similar to 4.6#19) A company manufactures and sells x cameras per week.

The weekly price-demand equation is $x + 30p = 9000$

The weekly cost equation is $C(x) = 90{,}000 + 30x$

(A) Find the price function, graph it, and determine its domain.

x is the Demand

p (small p) is the Price. (selling price per item)

Solve the equation $x + 30p = 9000$ for p

$$30p = 9000 - x$$

$$p = \frac{9000 - x}{30} = \frac{9000}{30} - \frac{x}{30}$$

$$p = 300 - \frac{x}{30}$$

$$p(x) = 300 - \frac{x}{30}$$

Graph $p(x) = 300 - \frac{x}{30}$

$y = mx + b$

$m = -\frac{1}{30}$

$b = 300$

$y = \left(-\frac{1}{30}\right)x + 300$

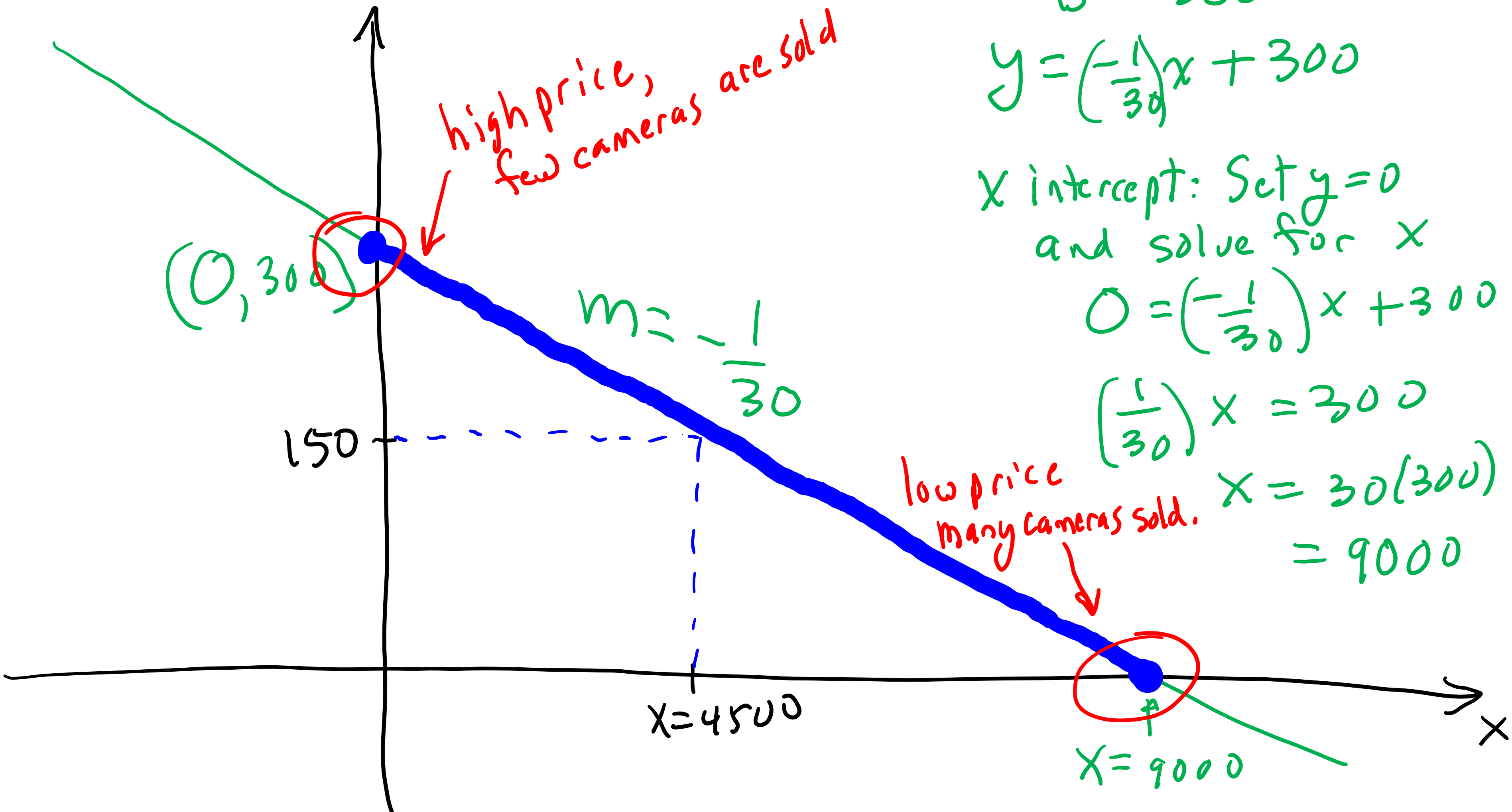
x intercept: Set $y = 0$ and solve for x

$0 = \left(-\frac{1}{30}\right)x + 300$

$\left(\frac{1}{30}\right)x = 300$

$x = 30(300)$

$= 9000$



must have $x \geq 0$ and $P \geq 0$

The domain of the price function is $0 \leq x \leq 9000$

(B) If the goal is to maximize the *weekly revenue*, what price should the company charge for the cameras, and how many cameras should be produced per week?

Find the Revenue function

Revenue = number of items sold • selling price per item

= Demand • Price

$R = x \cdot p$ ← small p

$$R(x) = x \cdot p(x) = x\left(300 - \frac{x}{30}\right)$$

$$= 300x - \frac{x^2}{30}$$

$$= 300x - \left(\frac{1}{30}\right)x^2$$

domain

$0 \le x \le 9000$

Strategy

- Find value of x that maximizes $R(x)$
- Find the corresponding value of p

Maximize $R(x) = 300x - \left(\frac{1}{30}\right)x^2$ on the domain $[0, 9000]$

Observe: Continuous function on a closed interval.

Use the closed interval method.

Find critical numbers for $R(x)$

$$R'(x) = \frac{d}{dx}\left(300x - \left(\frac{1}{30}\right)x^2\right) = 300(1) - \left(\frac{1}{30}\right)(2x)$$

$$= 300 - \frac{x}{15} = 300 - \left(\frac{1}{15}\right)x$$

R' is polynomial, so there are no x values that cause $R'(x)$ to not exist.

Set $R'(x) = 0$ and solve for x.

$$0 = 300 - \left(\frac{1}{15}\right)x$$

$$\left(\frac{1}{15}\right)x = 300$$

$$x = (300)(15) = 4500$$

list of important x values	Value of $R(x) = x\left(300 - \frac{x}{30}\right)$
$x=0$ endpoint	$R(0) = 0\left(300 - \frac{0}{30}\right) = 0 \cdot 300 = 0$
$x=4500$ critical	$R(4500) = 4500\left(300 - \frac{4500}{30}\right) = \cdots = 675{,}000$
$x=9000$ endpoint	$R(9000) = 9000\left(300 - \frac{9000}{30}\right) = 9000(300-300) = 9000(0) = 0$

To maximize the weekly revenue, the company should sell $x = 4500$ cameras per week.

The corresponding price is $p(x) = 300 - \frac{x}{30}$

$p(4500) = 300 - \frac{4500}{30} = 300 - 150 = 150$. Sell cameras for \$150 each

(C) What is the maximum possible *weekly revenue*?

$R(4500) = \$675{,}000$ per week.

(D) If the goal is to maximize the *weekly profit*, what price should the company charge for the cameras, and how many cameras should be produced per week? Use calculus methods.

Profit = Revenue − Cost

$P(x) = R(x) - C(x)$

$= \left(300x - \left(\frac{1}{30}\right)x^2\right) - (90{,}000 + 30x)$

$= \left(-\frac{1}{30}\right)x^2 + 270x - 90{,}000$

domain is $[0, 9000]$

Find value of x that maximizes $P(x)$.

Find Critical numbers for $P(x)$

Find $P'(x)$

Set $P'(x) = 0$

Solve for x

$$P'(x) = \frac{d}{dx}\left(\left(-\frac{1}{30}\right)x^2 + 270x - 90{,}000\right)$$

$$= -\left(\frac{1}{30}\right)(2x) + 270 - 0$$

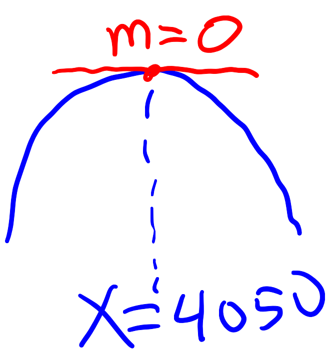
$$= -\left(\frac{1}{15}\right)x + 270$$

$$0 = P'(x) = -\left(\frac{1}{15}\right)x + 270$$

$$\left(\frac{1}{15}\right)x = 270$$

$$x = 270 \cdot 15 = 4050$$ critical number

Profit function $P(x)$ is a parabola facing down



So $x = 4050$ will be the location of the max. The company should sell 4050 cameras per week

The corresponding selling price is

$$p(x) = 300 - \frac{x}{30}$$

$$p(4050) = 300 - \frac{(4050)}{30} = 165$$

So to maximize profit, the company should charge \$165 per camera. They will sell 4050 cameras per week

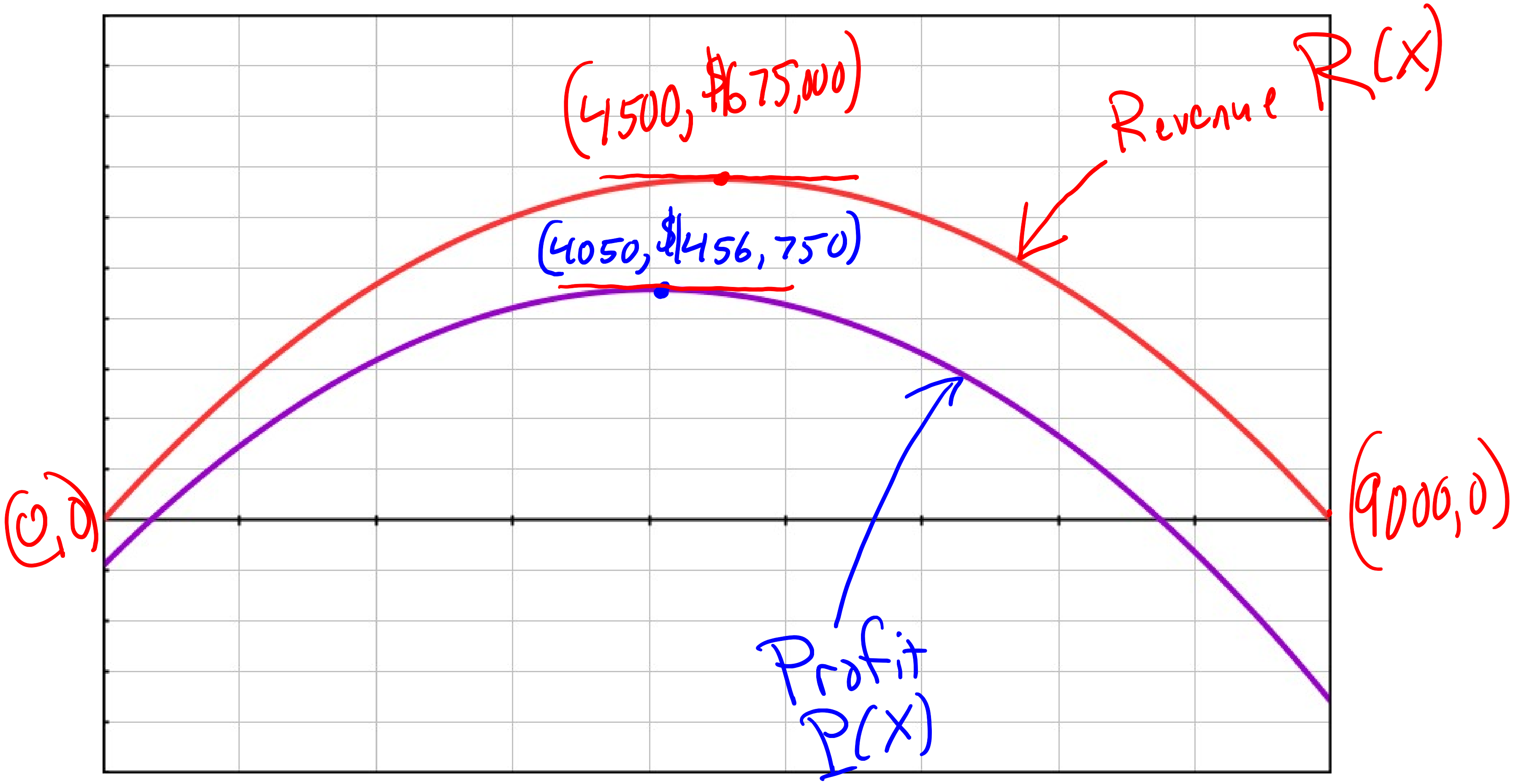
(E) What is the maximum *possible weekly profit*?

Profit $P(x) = -\left(\frac{1}{30}\right)x^2 + 270x - 90{,}000$

$P(4050) = -\left(\frac{1}{30}\right)(4050)^2 + 270(4050) - 90{,}000$

$= \cdots = \$456{,}750$

(F) Illustrate your results from **(B),(C),(D),(E)** on this given graph of $R(x)$ and $P(x)$



[Example 2] (similar to 4.6#25)

(A) A Coffee shop sells 1600 cups of coffee per day when price is $2.40 per cup.

What is the *daily revenue*?

Revenue = Demand × Price

$R = x \cdot p$

$= 1600 \cdot (2.40)$

$= \$3840$ per day

(B) A market survey predicts that for every \$0.05 price reduction, 50 more cups of coffee will be sold.

How much should the coffee shop charge per cup in order to maximize *daily revenue*?

How many cups will be sold? What will be the resulting *daily revenue*?

Solution Let n be the number of \$0.05 price reductions.

Then the selling price will be $p = 2.40 - .05n$

The number of cups sold will be

demand $x = 1600 + 50n$

The Revenue will be

$$\text{Revenue} = \text{demand} \cdot \text{price}$$

$$R = x \cdot p$$

$$R(n) = (1600 + 50n) \cdot (2.40 - .05n)$$

$$= -2.5n^2 + 40n + 3840$$

Find value of n that maximizes

$$R(n) = -2.5n^2 + 40n + 3840$$

Strategy
- Find $R'(n)$
- Set $R'(n) = 0$
- Solve for n

$$R'(n) = \frac{d}{dn}\left(-2.5n^2 + 40n + 3840\right)$$

$$= -2.5(2n) + 40(1) + 0$$

$$= -5n + 40$$

$$0 = R'(n) = -5n + 40$$

$$5n = 40$$

$$n = 8$$

So the selling price $p = 2.40 - .05(n)$

Should be $p = 2.40 - .05(8)$

$= 2.40 - .40$

$= \$2$ per cup

And the shop will sell $x = (1600 + 50n)$

$x = 1600 + 50(8)$

$= 1600 + 400$

$= 2000$ cups of coffee per day

The resulting Revenue $R = x \cdot p$ will be

$R = 2000 \cdot 2 = \$4000$ dollars per day.

The textbook uses a different approach, to this kind of problem. Here is what they would do.

They use variable x = number of \$0.05 price reductions.

Then $price = 2.40 - 0.05x$ and $demand = 1600 + 50x$.

$$Revenue = demand \cdot price = (1600 + 50x) \cdot (2.40 - 0.05x) = -2.5x^2 + 40x + 3840$$

It may be nice to have x as the variable,

but it is confusing, because we no longer have $R = x \cdot p$