Subject for this video:

Two Variable Abstract Max Min Problems

Reading: 4.6 Optimization

- **General:** Section ~~4.5~~ ~~Absolute Maxima and Minima~~
- **More Specifically:** there is no discussion of this kind of problem in the reading

Homework: H65: Two Variable Abstract Max Min Problems (4.6#9,13,15,17)

Recall the definition of Critical Numbers from Section 4.1 (introduced in the Video for H55)

Definition of *Critical Number for* $f(x)$

Words: *critical number for* $f(x)$

Meaning: a number $x = c$ that satisfies these two requirements:

- The number $x = c$ is a partition number for $f'(x)$.
- The number $x = c$ is in the domain of $f(x)$.

That is,

- $f'(c) = 0$ or $f'(c)$ does not exist
- $f(c)$ exists

Recall the definition Absolute Extrema from Section 4.5 (introduced in the Video for H61)

DEFINITION Absolute Maxima and Minima

If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute maximum** of f. If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute minimum** of f. An absolute maximum or absolute minimum is called an **absolute extremum.**

And these Theorems about absolute extrema from Section 4.5 (introduced the Video for H62)

There is one important situation where *both absolute max and absolute min are guaranteed*.

THEOREM 1 Extreme Value Theorem

A function f that is continuous on a closed interval $[a, b]$ has both an absolute maximum and an absolute minimum on that interval.

And there is a theorem that tells us where Absolute Extrema *have to occur*.

THEOREM 2 Locating Absolute Extrema

Absolute extrema (if they exist) must occur at critical numbers or at endpoints.

Theorems 1 and 2 are the basis for the following *procedure* (the ***Closed Interval Method***) for finding the *absolute extrema* on a *closed interval* for a function that is *continuous* on that interval. This procedure was discussed in the Video for Homework H62.

PROCEDURE Finding Absolute Extrema on a Closed Interval

Step 1 Check to make certain that f is continuous over $[a, b]$.
Step 2 Find the critical numbers in the interval (a, b).
Step 3 Evaluate f at the endpoints a and b and at the critical numbers found in step 2.
Step 4 The absolute maximum of f on $[a, b]$ is the largest value found in step 3.
Step 5 The absolute minimum of f on $[a, b]$ is the smallest value found in step 3.

But what about the situation where the domain of the function is *not* a closed interval? How does one determine the absolute extrema that *do* occur? As we will saw in the video for Homework H63, that question is answered in different ways for different functions.

For some familiar function types, the approach can be to

- First, consider the end behavior to determine which kinds of absolute extrema will occur.
- Then, find the locations of those extrema in the following way:
    - Find the critical numbers of the function in the domain
    - Compute values of $f(x)$ at those critical numbers and at endpoints (if there are any)
    - Identify the absolute max or min values that you know will occur.

For functions that are not familiar function types, the approach is to

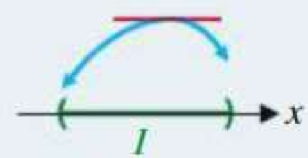
- Find the critical numbers of the function in the domain.
- Determine if any of those critical numbers is the location of an absolute extremum by either
    - studying the sign behavior of $f'(x)$ to determine increasing/decreasing behavior of $f(x)$
    - or using the Second Derivative Test for Absolute Extrema on an Interval

THEOREM 3 Second-Derivative Test for Absolute Extrema on an Interval

Let f be continuous on an interval I from a to b with only one critical number c in (a, b).

If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on I.

If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on I.



In Section 4.6, we are studying problems involving Optimization.

Optimization problems are simply Max/Min problems, but they may have complications

- They may be presented as word problems, about applications to real world situations.
- You will probably have to figure out the function and its domain
- They may have domains that are not closed intervals
- They may involve more than one variable

Homework H65 consists of

Two Variable Abstract Optimization Problems

[Example 1] (Similar to 4.6#9) Find positive numbers x, y such that

- The sum $2x + y = 900$.
- The product maximized. the product is maximized.

Solution:

(Step 1) Identify Equation I: $2x+y=900$

(Step 2) Write Equation II involving x and y and the letter P for the product.

$P = xy$ maximize P

must have $x > 0$ and $y > 0$

(Step 3) Solve Equation I for y in terms of x.

Equation I $\quad 2x + y = 900$

$y = 900 - 2x$

new equation I

(Step 4) Substitute New Equation I into Equation II and simplify to get a new equation that gives the product ~~x~~ P as a function of just one variable x. Call this function $P(x)$. Determine the domain of this function.

New Equation I: $y = 900 - 2x$

Equation II: $P = x \cdot y$

Substitute I into II

$$P = x \cdot (900 - 2x) = 900x - 2x^2$$

$$P(x) = 90x - 200x^2$$

Domain: We know $x > 0$

We also know $y > 0$. So $900 - 2x > 0$

$$900 > 2x$$

$$450 > x$$

Conclude that the domain is $0 < x < 450$, or $(0, 450)$

(Step 5) Using Calculus, find the value of x that maximizes $P(x)$.

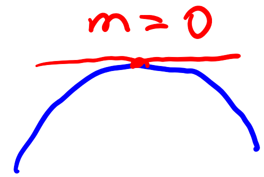
Goal: Find the value of x that maximizes

$P(x) = 900x - 2x^2$ on the domain $(0, 450)$

Observe graph of $P(x)$ will be a parabola facing down.

It will have a max at the one point

where $p'(x) = 0$



Strategy: • Find $P'(x)$

• Set $p'(x) = 0$

• Solve for x

$$P'(x) = \frac{d}{dx}\left(900x - 2x^2\right) = 900 - 2(2x)$$

$$= 900 - 4x$$

$$0 = P'(x) = 900 - 4x$$

$4x = 900$ $\qquad x = 225$

(Step 6) Find corresponding values of y and the product.

$$y = 900 - 2x = 900 - 2(225) = 900 - 450$$

$$y = 450$$

Product $P = x \cdot y = 225(450) = \cdots = 101{,}250$

[Example 2] (similar to 4.6#13) Find positive numbers x, y such that

- The product is 9000.
- The sum $10x + 25y$ is minimized.

$x > 0$

$y > 0$

Solution:

(step 1) Write an Equation I involving x and y expressing the fact that the product is 9000:

Equation I: $x \cdot y = 9000$

(step 2) Write an equation II involving x and y and the letter S for sum:

Equation II: $S = 10x + 25y$

minimize S.

(step 3) Solve Equation I for y in terms of x. New Equation I.

Equation I $\quad x \cdot y = 9000$

New Equation I $\quad y = \frac{9000}{x}$

(step 4) Substitute Equation I into Equation II and simplify to get a new equation that gives the sum S as a function of just one variable x. Call this function $S(x)$. Find its domain.

Now Equation I: $y = \frac{9000}{x}$

Equation II: $S = 10x + 25y$

Substitute I into II: $S = 10x + 25\left(\frac{9000}{x}\right)$

$$S(x) = 10x + \frac{25(9000)}{x}$$

Domain We know $x > 0$

But $y > 0$ is also required.

But $y = \frac{9000}{x}$ so if $x > 0$ then y will automatically be > 0

So the domain is $x > 0$. That is, $(0, \infty)$

(step 5) Using calculus, find the value of x that minimizes $S(x)$.

Minimize $S(x) = 10x + \frac{25(9000)}{x}$ on the interval $(0,\infty)$

Strategy: Find critical numbers for $S(x)$

First convert $S(x)$ to power function form

$S(x) = \underbrace{10x + \frac{25(9000)}{x}}_{\text{positive exponent form}} = \underbrace{10x + 25(9000)x^{-1}}_{\text{power function form}}$

So $S'(x) = \frac{d}{dx}\left(10x + 25(9000)x^{-1}\right) = 10(1) + 25(9000)(-1x^{-1-1})$

$= \underbrace{10 - 25(9000)x^{-2}}_{\text{power function form}} = \underbrace{10 - \frac{25(9000)}{x^2}}_{\text{positive exponent form}}$

Partition numbers for $S'(x) = 10 - \frac{25(9000)}{x^2}$

Observe $S'(0) = 10 - \frac{25(9000)}{0^2}$ Does not exist.

So $x=0$ is a partition number for $S'(x)$

Look for x values that cause $S'(x) = 0$

$$0 = 10 - \frac{25(9000)}{x^2}$$

$$\frac{25(9000)}{x^2} = 10$$

$$x^2 = \frac{25(9000)}{10} = 25(900)$$

$$x = \pm\sqrt{25(900)} = \pm\sqrt{25}\sqrt{900} = \pm 5(30)$$

$$= \pm 150$$

$x = -150$, $x = 150$

So the partition numbers for $S'(x)$ are

$x = 0$ because $S'(0)$ DNE

$x = -150$ because $S'(-150) = 0$

$x = 150$ because $S'(150) = 0$

See if they are critical numbers for $S(x) = 10x + \frac{25(9000)}{x}$

$$S(0) = 10(0) + \frac{25(9000)}{0} \text{ DNE}$$

$$S(150) = 10(150) + \frac{25(9000)}{150} = \cdots = 3000$$

$$S(-150) = 10(-150) + \frac{25(9000)}{-150} = \cdots = -3000.$$

The only critical numbers for $S(x)$ are $x = -150$, $x = 150$

There is only one critical number in the interval $(0, \infty)$. It is $x = 150$.

Find $S''(150)$ in order to use 2nd Derivative test

We found $S'(x) = 10 - 25(9000)x^{-2}$

So $S''(x) = \frac{d}{dx}\left(10 - 25(9000)x^{-2}\right)$

$= 0 - 25(9000)\left(-2x^{-2-1}\right)$

$= -25(9000)(-2)x^{-3}$

$= \frac{25(9000)(2)}{x^3}$

$S''(150) = \frac{25(9000)(2)}{(150)^3} > 0$

So $S'(150)=0$ and $S''(150)>0$ and $x=150$ is the only critical number on the interval $(0,\infty)$

2nd Derivative Test tells us that $x=150$ is the location of the absolute min (step 6) Find the corresponding values of y and sum, S.

$$y = \frac{9000}{x}$$

$$y = \frac{9000}{150} = 60$$

$$\text{Sum } S = 10x + 25\left(\frac{9000}{x}\right)$$

$$S = 10(150) + 25\left(\frac{9000}{150}\right)$$

$$= 10(150) + 25(60)$$

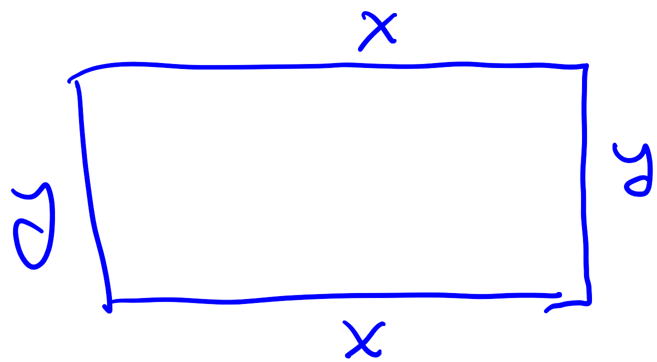
$$= 1500 + 1500$$

$$= 3000$$

Conclusion $x = 150$, $y = 60$, $S = 3000$

[Example 3] (Similar to 4.6#17) Find the dimensions of a rectangle with a *perimeter* of 150 feet that has the maximum *area*.

Solution



Steps 1&2

Perimeter $= 150 = x + x + y + y = 2x + 2y$

Area $A = x \cdot y$ maximize area

Two equations: Equation I $150 = 2x + 2y$

Equation II $A = xy$ maximize A

Step 3 Solve Equation I for y

$$150 = 2x + 2y$$

$$75 = x + y$$

$$y = 75 - x$$

Step 4 Substitute Equation I into Equation II

$$A = x \cdot y = x \cdot (75 - x) = 75x - x^2$$

$$A(x) = 75x - x^2$$

Domain: clearly $x > 0$ (length can't be 0 or negative)

and $y > 0$

but $y = 75 - x$

So we must have $75 - x > 0$

$$75 > x$$

So the domain is $0 < x < 75$

So our job is to maximize $A(x) = 75x - x^2$

on the domain $(0, 75)$

Step 5 Using calculus, find the value of x

Strategy: Set $A'(x) = 0$ and solve for x.

$$A(x) = 75x - x^2$$

$$A'(x) = 75 - 2x = 0$$

$$2x = 75$$

$$x = \frac{75}{2}$$

Observe $x = \frac{75}{2}$ is in the interval $(0, 75)$

and $A(x)$ is a parabola facing down.

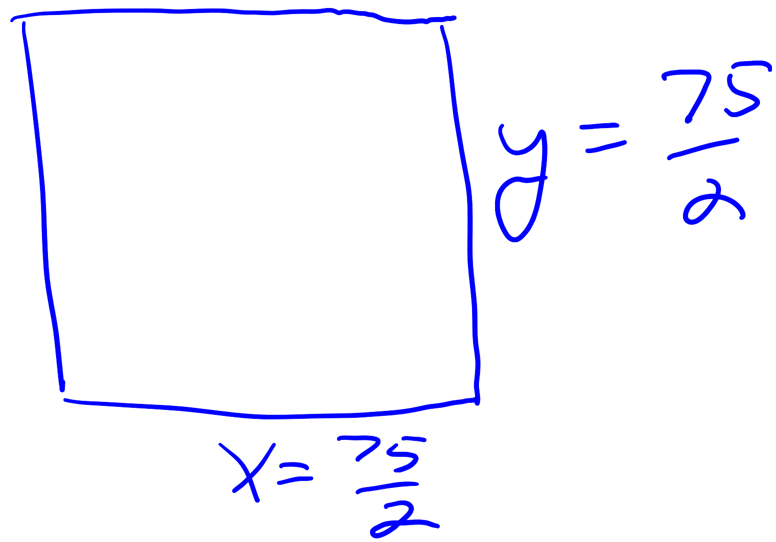
So $x = \frac{75}{2}$ must be the location of the max.

The corresponding value of y is

$$y = 75 - x = 75 - \frac{75}{2} = \frac{75}{2}$$

Conclusion

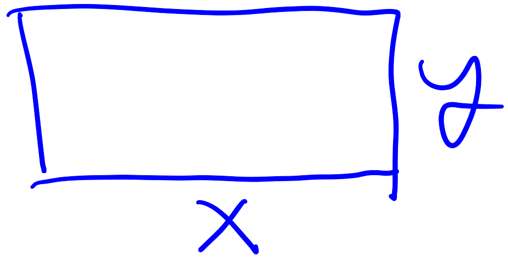
For a rectangle with perimeter 75 to have maximum area, its dimensions should be $x = \frac{75}{2}$, $y = \frac{75}{2}$



A square!

[Example 4] (Similar to 4.6#15) Find the dimensions of a rectangle with an *area* of 150 square feet that has the minimum *perimeter*.

Solution



Area $= xy = 150$

Perimeter

$P = 2x + 2y$ minimize P

Step 1 Equation I is $xy = 150$

Step 2 Equation II is $2x + 2y = P$ minimize P

Step 3 Solve equation I for y

$$y = \frac{150}{x}$$

Step 4 Substitute into equation 2

$$P = 2x + 2y = 2x + 2\left(\frac{150}{x}\right)$$

$$P(x) = 2x + \frac{300}{x}$$ Domain $x > 0$. That is $(0, \infty)$

Step 5 Using calculus, find value of x that minimizes $P(x) = 2x + \frac{300}{x}$ on the interval $(0, \infty)$

Rewrite $P(x)$ to make derivative easier

$$P(x) = 2x + \frac{300}{x} = 2x + 300x^{-1}$$

$$P'(x) = \frac{d}{dx}\left(2x + 300x^{-1}\right) = 2(1) + 300\left(-1x^{-1-1}\right)$$

$$P'(x) = 2 - 300x^{-2} = 2 - \frac{300}{x^2}$$

Partition numbers for $P'(x)$

$x = 0$ because $P'(0)$ DNE,

Set $P'(x) = 0$ and solve for x

$$0 = 2 - \frac{300}{x^2}$$

$$\frac{300}{x^2} = 2$$

$$x^2 = \frac{300}{2} = 150$$

$$x = \pm\sqrt{150} = \pm\sqrt{25 \cdot 6} =$$

$$= \pm\sqrt{25} \cdot \sqrt{6} = \pm 5\sqrt{6}$$

Partition numbers for $P'(x)$ are

$x = 0$, $x = -5\sqrt{6}$, $x = 5\sqrt{6}$

Observe $P(0)$ DNE, so the critical numbers for $P(x)$ are $x = -5\sqrt{6}$, $x = 5\sqrt{6}$

So we are to minimize $P(x) = 20x + \frac{300}{x}$ on the interval $(0, \infty)$,

and there is only one critical number $x = 5\sqrt{6}$ on that interval. So $x = 5\sqrt{6}$ must be the location of the min.

Check with 2nd Derivative test

$$P'(x) = 2 - 300x^{-2} = 2 - \frac{300}{x^2}$$

So $P''(x) = 0 - 300(-2x^{-2-1}) = 600x^{-3} = \frac{600}{x^3}$

$$P''(5\sqrt{6}) = \frac{600}{(5\sqrt{6})^3} > 0$$

Since $P'(5\sqrt{6}) = 0$ and $P''(5\sqrt{6}) > 0$, we conclude that $x = 5\sqrt{6}$ must be the location of an absolute min.

The corresponding value of y is

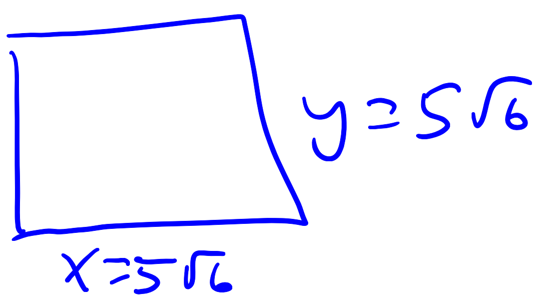
$$y = \frac{150}{x}$$

$$y = \frac{150}{5\sqrt{6}}$$

$$= \frac{30}{\sqrt{6}}$$

$$= \frac{5 \cdot 6}{\sqrt{6}} = 5\sqrt{6}$$

So $x = 5\sqrt{6}$ and $y = 5\sqrt{6}$



Square!