Subject for this video:

Basic Indefinite Integrals

Reading:

- **General:** Section 5.1 Antiderivatives and Indefinite Integrals
- **More Specifically:** Pages 324 - 330, Examples 2,3

Homework: H70: Basic Indefinite Integrals (5.1#9,11,13,17,19,21,23)

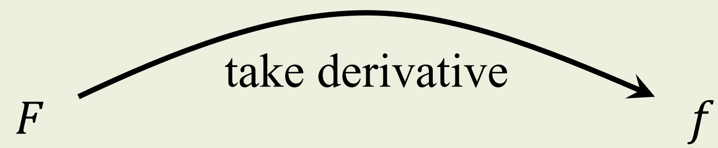
Recall the definition of *antiderivative* from the previous video.

Definition of Antiderivative

Words: *F is an antiderivative of f.*

Meaning: f is the derivative of F. That is, $f = F'$.

Arrow diagram:



Common Form of Question:

Is One Given Function an Antiderivative of Another Given Function?

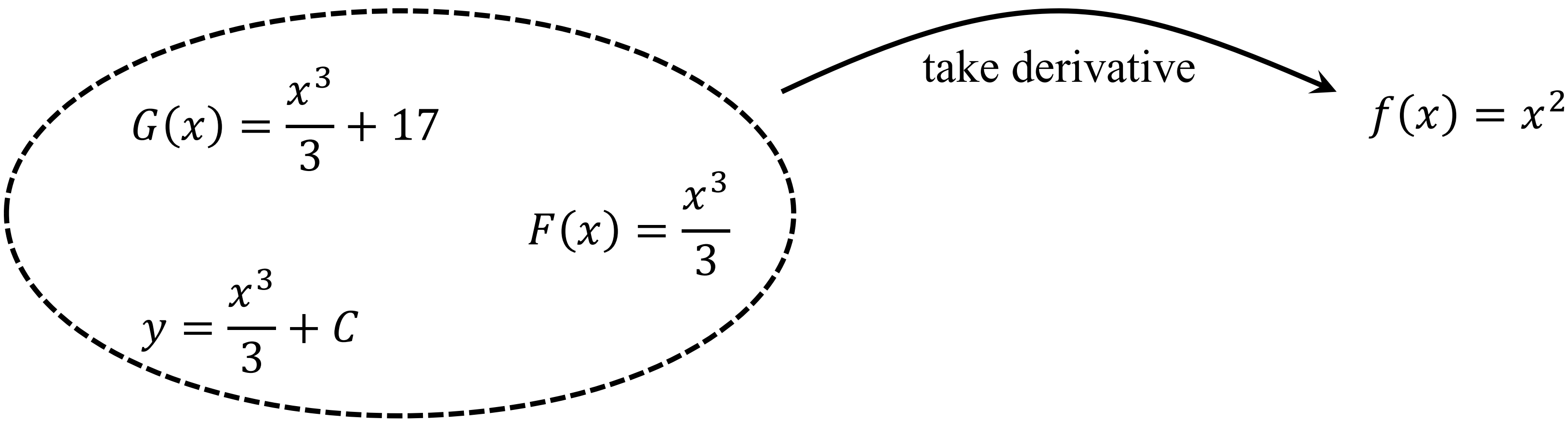
That is, given a function F and a function f, is F an antiderivative of f?

To answer this question, simply find the derivative of F.

- If $F' = f$, then F is an antiderivative of f.
- If $F' \neq f$, then F is not an antiderivative of f.

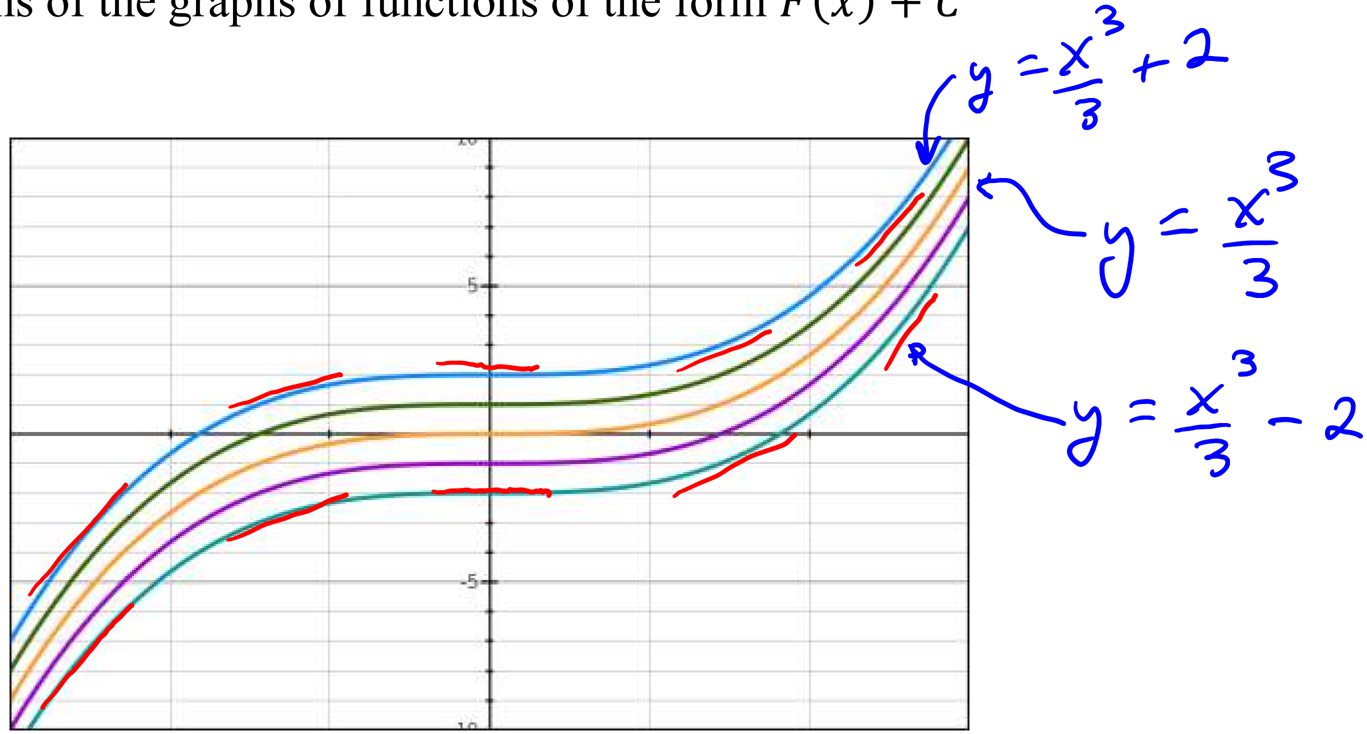
Write a clear conclusion.

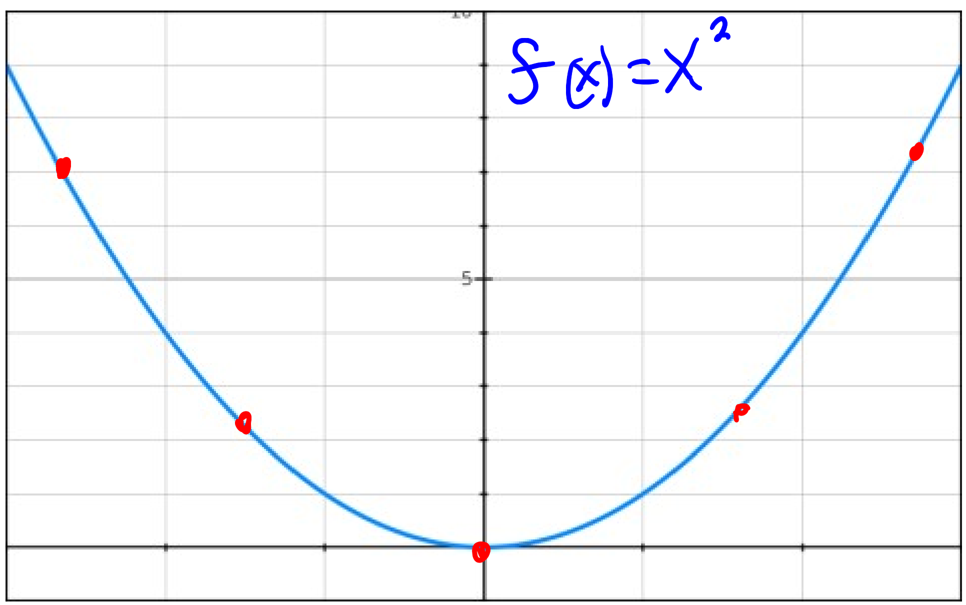
Also recall, from the previous video, this diagram illustrating the collection of all antiderivatives of the function $f(x) = x^2$.



the collection of all antiderivatives of $f(x) = x^2$

This makes sense in terms of the graphs of functions of the form $F(x) + C$





We discussed how the observations about the function $f(x) = x^2$ and its antiderivatives can be generalized to other functions.

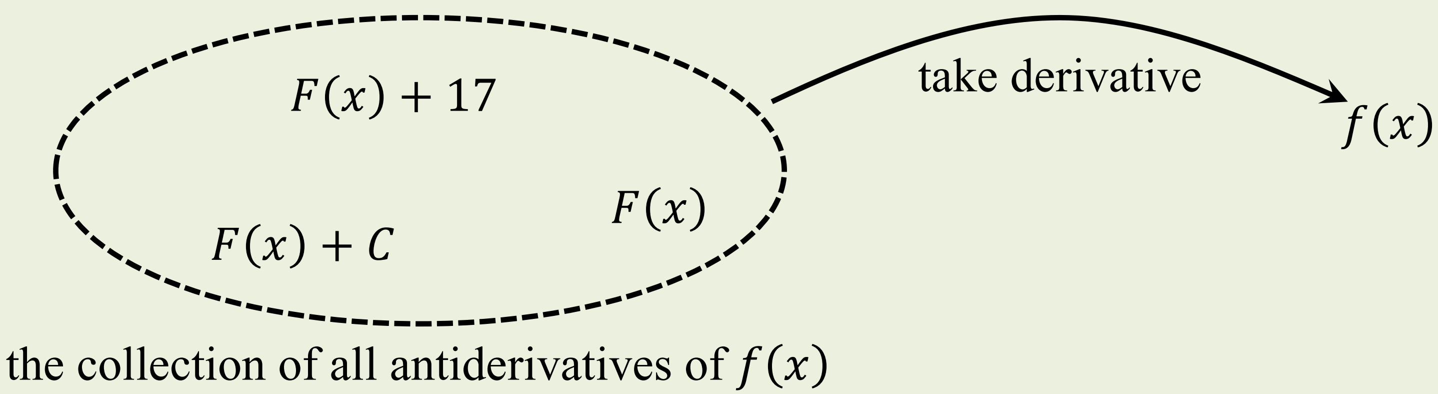
Theorem about the Collection of Antiderivatives of a Function

If a function $F(x)$ is an antiderivative of a function $f(x)$, then any function of the form

$$F(x) + C$$

where C is a real number constant, will also be an antiderivative of $f(x)$.

Furthermore, these are *all* the antiderivatives of $f(x)$. That is, if $G(x)$ is also an antiderivative of $f(x)$, then it must be that $G(x) = F(x) + C$ where C is a real number constant.



The previous video ended with a discussion of

Particular Antiderivative and General Antiderivative

When a choice of an *actual number* for C is made, the resulting function is called a ***particular antiderivative*** of $f(x)$. That is,

- The function $F(x) = \frac{x^3}{3}$ is a *particular antiderivative* of $f(x) = \frac{x^3}{3}$.
- The function $G(x) = \frac{x^3}{3} + 17$ is a *particular antiderivative* of $f(x)$.

But if C has not been chosen, then the *function form*

$$y = \frac{x^3}{3} + C$$

is called ***the general antiderivative*** of $f(x)$. Note the use of the word ***the***. Realize that this expression is a *function form*. There is only one such form.

In this video, we will discuss Indefinite Integrals, which are defined as follows.

Definition of Indefinite Integral

symbol: $\int f(x)dx$

spoken: the ***indefinite integral*** of $f(x)$

meaning: the ***general antiderivative*** of $f(x)$

Remark: We know that, given one function $F(x)$ that is known to be an antiderivative of $f(x)$, we can get all other antiderivatives by adding constants to $F(x)$. The *general antiderivative* of $f(x)$ is denoted by writing $F(x) + C$, where C is a constant that can be any real number. That is,

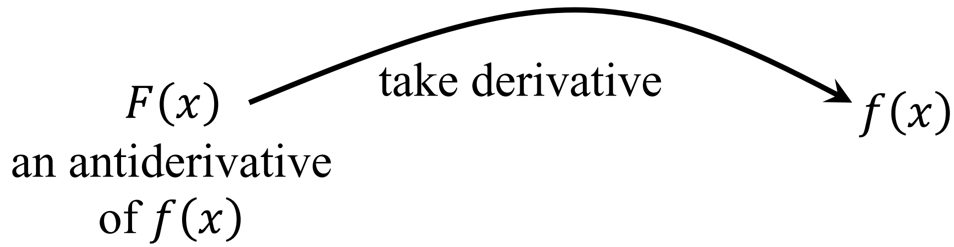
$$\text{If } F'(x) = f(x) \text{ then } \int f(x)dx = F(x) + C$$

Additional Terminology:

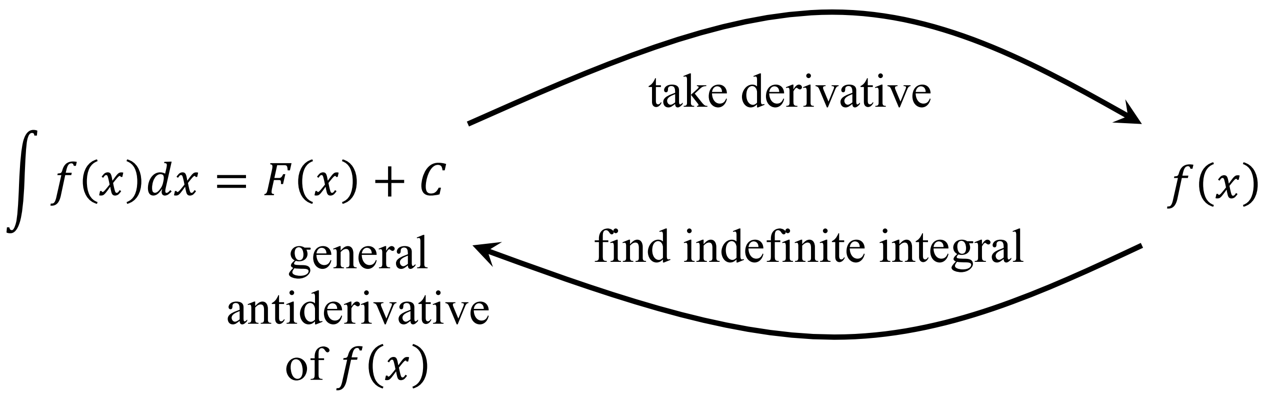
- The function $f(x)$ inside the integral symbol is called the ***integrand***.
- The unknown constant C is called the ***constant of integration***.

The *indefinite integral* can be illustrated by the diagrams below.

Suppose:



Then:



Remark on Wording:

Notice the above definition of indefinite integral says ***the*** *indefinite integral of* $f(x)$.

Contrast this with the earlier definition of antiderivative that said ***an*** *antiderivative of* f.

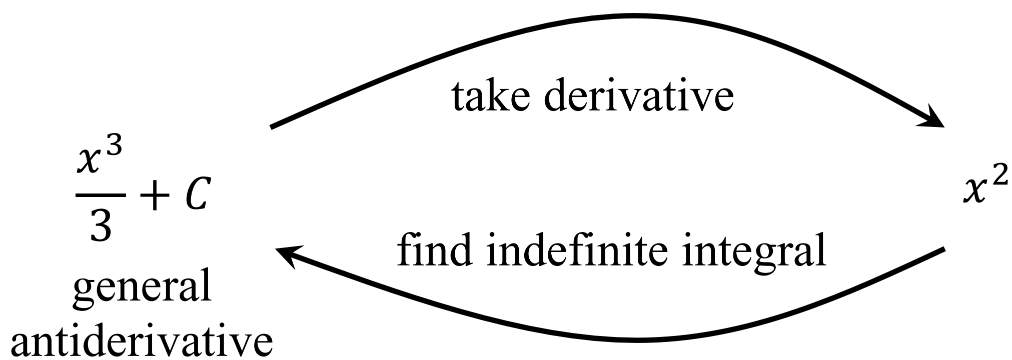
There are ***many*** antiderivatives of $f(x)$, but there is ***only one*** indefinite integral of $f(x)$.

In today's video, we will learn rules for finding indefinite integrals. First, though, we will simply revisit some of our examples of *antiderivatives* from previous videos, and and rewrite the results using the new terminology and notation of *indefinite integrals*.

In the videos for Homeworks H68 and H69, we found that $F(x) = \frac{x^3}{3}$ and $G(x) = \frac{x^3}{3} + 17$ were both antiderivatives of $f(x) = x^2$. They are ***particular antiderivatives*** of $f(x) = x^2$. Using integral notation and terminology, we write

$$\int x^2 dx = \frac{x^3}{3} + C$$

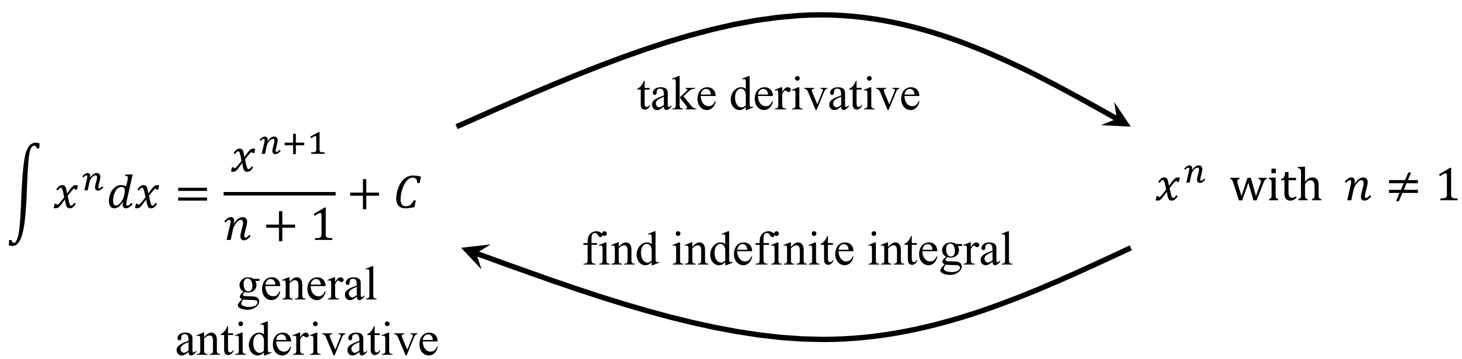
That is, the ***general antiderivative*** of x^2 is $\frac{x^3}{3} + C$



In the video for Homework H68, we found that for all real numbers $n \neq -1$, $F(x) = \frac{x^{n+1}}{n+1}$ is an antiderivative of $f(x) = x^n$. Using integral notation and terminology, we write the following, which will be our first indefinite integral rule

The *power rule for indefinite integrals*: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$

That is, when $n \neq -1$,the ***general antiderivative*** of x^n is $\frac{x^{n+1}}{n+1} + C$



In another example in the video for Homework H68, we found that $F(x) = x\ln(x) - x$ is an antiderivative of $f(x) = \ln(x)$. Using integral notation and terminology, we write the following, which will be another indefinite integral rule

The $\ln(x)$ *rule for indefinite integrals*: $\int \ln(x)\,dx = x\ln(x) - x + C$

That is, the ***general antiderivative*** of $\ln(x)$ is $x\ln(x) - x + C$

take derivative

$\int \ln(x)\,dx = x\ln(x) - x + C$ $\qquad\qquad$ $\ln(x)$

general antiderivative

find indefinite integral

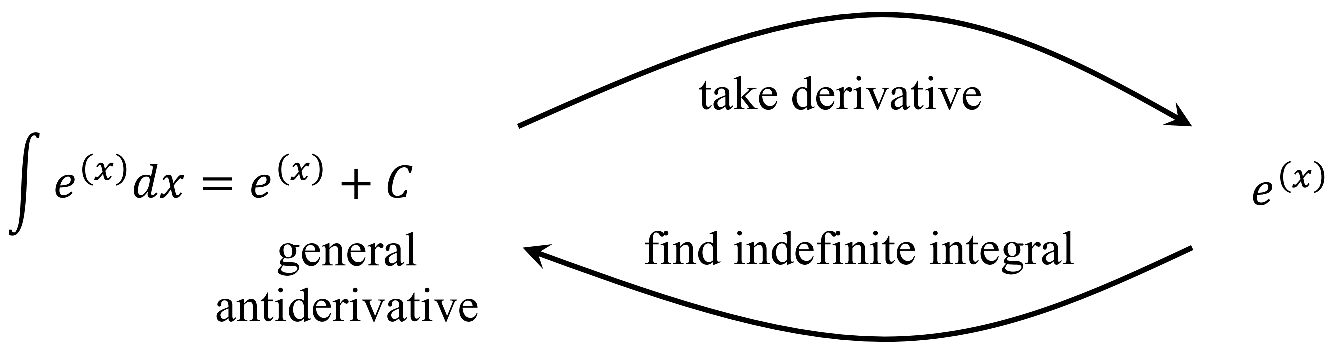
In another example in the video for Homework H68, we found that $g(x) = 5e^{(x)}$ is an antiderivative of itself. A simpler result would be the following:

The function $e^{(x)}$ is an antiderivative of itself.

Using integral notation and terminology, we have the following indefinite integral rule

The $\boldsymbol{e^{(x)}}$ *rule for indefinite integrals*: $\int e^{(x)}dx = e^{(x)} + C$

That is, the ***general antiderivative*** of $e^{(x)}$ is $e^{(x)} + C$



Finding $\int \frac{1}{x}\,dx$

If we tried to find this indefinite integral using the power rule, we would find

Rewrite integrand in power function form $\frac{1}{x} = x^{-1}$

$$\int \frac{1}{x}\,dx = \int x^{-1} = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C = \frac{1}{0} + C \quad \text{undefined!}$$

$n = -1$

Power Rule with $n = -1$

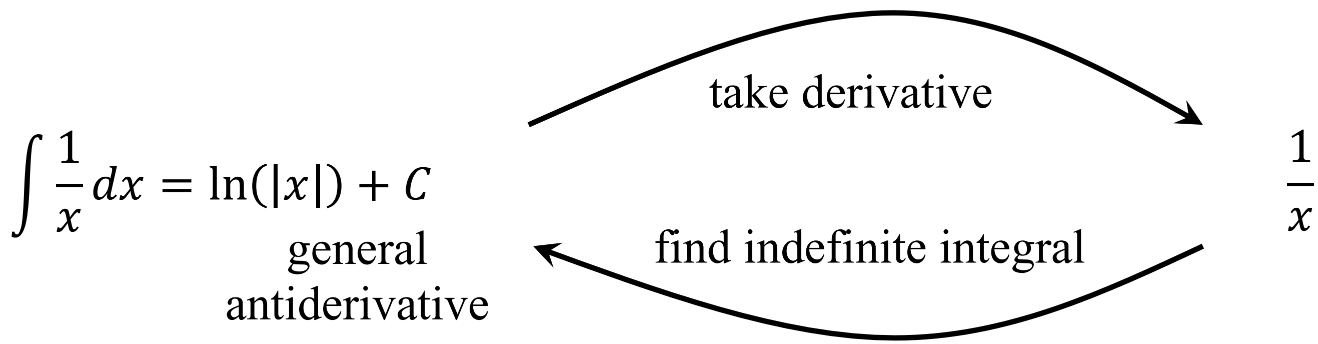
Invalid method (and incorrect result)

But remember, the power rule does not apply to the case of a power function with $n = -1$.

So how, then, *should* we find the integral?

Here is the rule

The $\boldsymbol{\frac{1}{x}}$ *rule for indefinite integrals*: $\int \frac{1}{x}\,dx = \ln(|x|) + C$ for all $x \neq 0$



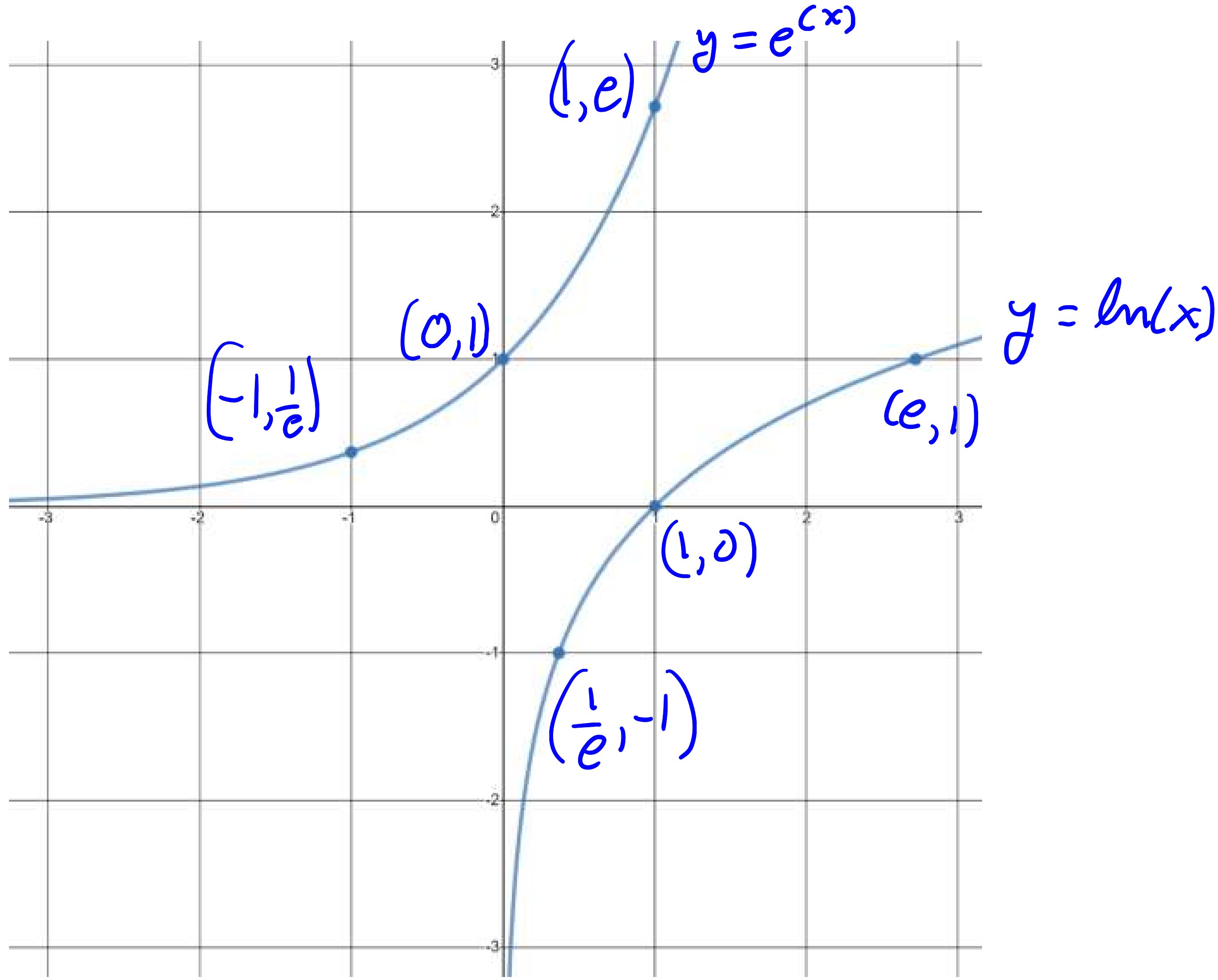
To understand this integral rule, we need to understand the corresponding derivative relationship:

Two equation form: If $f(x) = \ln(|x|)$ then $f'(x) = \frac{1}{x}$

Single equation form: $\frac{d}{dx}\ln(|x|) = \frac{1}{x}$

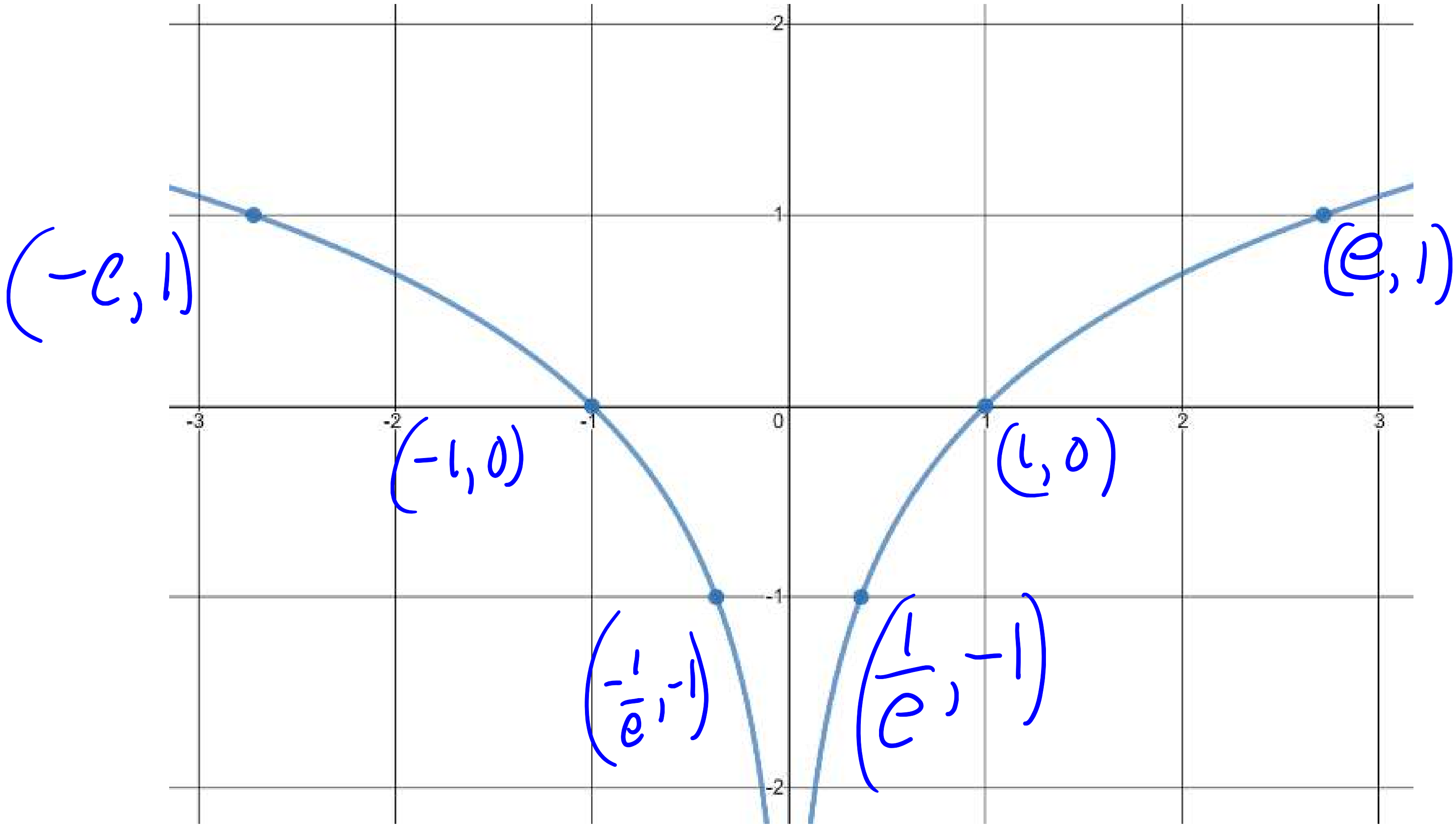
To understand this derivative relationship, we should examine the graphs of $y = \ln(|x|)$ and $y = \frac{1}{x}$

The graph of $y = \ln(x)$ is obtained from the graph of $y = e^{(x)}$ by interchanging all the x, y values.



Note that the domain of $y = \ln(x)$ is the set of all $x > 0$. That is, the interval $(0, \infty)$

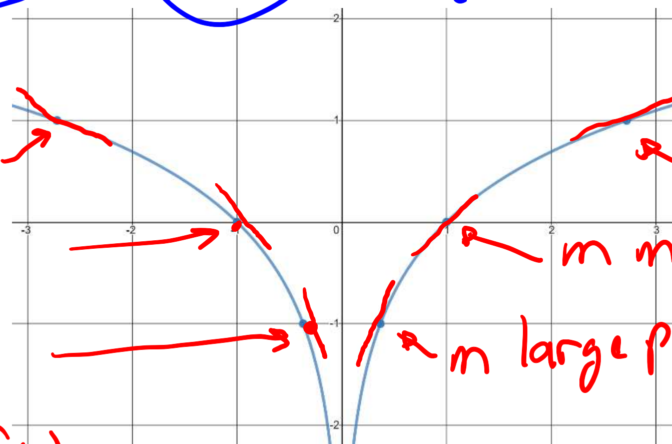
The graph of $y = \ln(|x|)$ ~~has~~ is as shown.



Note that the domain of $y = \ln(|x|)$ is the set of all $x \neq 0$.

Now consider the graphs of $\ln(|x|)$ and $y = \frac{1}{x}$

$f(x) = \ln(|x|)$



m small negative

m medium negative

m large negative

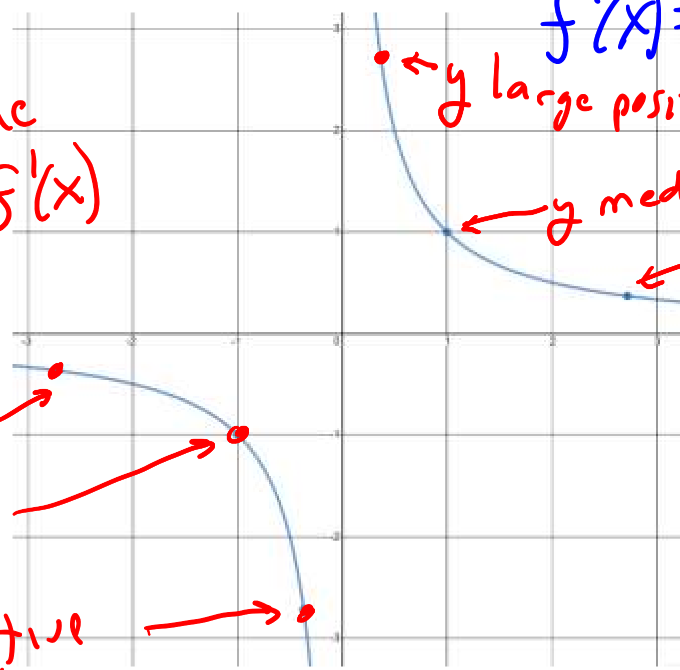
m small positive

m medium positive

m large positive

The numbers that are the slopes of the tangent lines on $f(x)$ equal

$f'(x) = \frac{1}{x}$



the numbers that are the y values on the graph of $f'(x)$

y large positive

y medium positive

y small positive

y small negative

y medium negative

y large negative

So, it is believable that if $f(x) = \ln(|x|)$ then $f'(x) = \frac{1}{x}$

And therefore, the corresponding indefinite integral rule makes sense.

$$\int \frac{1}{x} dx = \ln(|x|) + C \text{ for all } x \neq 0$$

Recall the ***Constant Multiple Rule for Derivatives***:

If $f(x)$ is a function and a is a real number, then

$$\frac{d}{dx}af(x) = a\frac{d}{dx}f(x)$$

The corresponding rule for integrals is as follows

Constant Multiple Rule for Indefinite Integrals

If $f(x)$ is a function and a is a real number, then

$$\int af(x)dx = a\int f(x)dx$$

This rule seems obvious enough. But we will see that there is a bit of subtlety in the issue of the constant of integration. That subtlety will be discussed in an example.

Here is a summary of the Indefinite Integral Rules that we have discussed in this video

The *power rule*:	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$
The $\frac{1}{x}$ *rule*:	$\int \frac{1}{x} dx = \ln(\lvert x \rvert) + C$ for all $x \neq 0$
The $e^{(x)}$ *rule*:	$\int e^{(x)} dx = e^{(x)} + C$
The $\ln(x)$ *rule*:	$\int \ln(x)\, dx = x \ln(x) - x + C$
The *constant multiple rule*:	$\int af(x)dx = a \int f(x)dx$

[Example 1] Find the following indefinite integrals.

$(\boldsymbol{A})\ \int x^2 dx$ = $\frac{x^{2+1}}{2+1} + C = \frac{x^3}{3} + C$

($n=2$; power rule with $n=2$)

$(\boldsymbol{B})\ \int x^8 dx$ = $\frac{x^{8+1}}{8+1} + C = \frac{x^9}{9} + C$

($n=8$; power rule with $n=8$.)

$(C)\ \int x\,dx$

Observe that $x = x^1$

$$\int x\,dx = \int x^1\,dx = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$$

← n = 1

↑ power rule with n = 1

$(D)\ \int 1\,dx$

Observe that $1 = x^0$

$$\int 1\,dx = \int x^0\,dx = \frac{x^{0+1}}{0+1} + C = \frac{x^1}{1} + C = x + C.$$

← n = 0

↑ use power rule with n = 0

Corresponding derivative equation: $\frac{d}{dx}(x + C) = 1 + 0 = 1$

(E) $\int \frac{1}{x^8}\,dx$

Put integrand in power function form

$$\frac{1}{x^8} = x^{-8}$$

$n = -8$

$$\int \frac{1}{x^8}\,dx = \int x^{-8}\,dx = \frac{x^{-8+1}}{-8+1} + C = \frac{x^{-7}}{-7} + C = -\frac{1}{7x^7} + C$$

power rule with $n = -8$

power function form

positive exponent form

(F) $\int 5x^8dx = 5\int x^8 dx = 5\left(\frac{x^{8+1}}{8+1} + C\right) =$

$n = 8$

constant multiple rule

power rule with $n = 8$

$$= 5\left(\frac{x^9}{9} + C\right)$$

$$= \frac{5x^9}{9} + 5C$$

The number C can be any real number.

Then $5C$ can be any real number.

$$= \frac{5x^9}{9} + D$$

where D can be any real number

(**G**) $\int \frac{5}{x^8}\,dx$

Write the integrand in power function form $\frac{5}{x^8} = 5x^{-8}$

$$\int \frac{5}{x^8}\,dx = \int 5x^{-8}\,dx = 5\int x^{-8}\,dx = 5\left(\frac{x^{-8+1}}{-8+1} + C\right)$$

$n=-8$

power rule with $n=-8$

$$= 5\left(\frac{x^{-7}}{-7} + C\right) = -\frac{5}{7}x^{-7} + 5C$$

$$= -\frac{5}{7x^7} + D$$

(**H**) $\int \frac{5}{x}\,dx$

Note we do not want to rewrite $\frac{5}{x} = 5x^{-1}$ (power function form)

because we can't use the power rule when $n = -1$.

Better to rewrite the integrand by separating the constant.

$$\frac{5}{x} = 5 \cdot \frac{1}{x}$$

$$\int \frac{5}{x}\,dx = \int 5 \cdot \frac{1}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\left(\ln(|x|) + C\right)$$

(constant multiple rule) (1/x rule)

$$= 5\ln(|x|) + 5C$$

$$= 5\ln(|x|) + D$$

(I) $\int 5\sqrt{x}\,dx$

Rewrite the integrand $5\sqrt{x} = 5x^{\frac{1}{2}}$

radical form

power function form

$$\int 5\sqrt{x}\,dx = \int 5x^{\frac{1}{2}}\,dx = 5\int x^{\frac{1}{2}}\,dx = 5\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C\right) =$$

Constant multiple rule

power rule with $n=\frac{1}{2}$

$$= 5\left(\frac{x^{3/2}}{3/2} + C\right) = 5\left(\frac{2x^{3/2}}{3} + C\right)$$

$$= \frac{10x^{3/2}}{3} + 5C$$

$$= \frac{10x^{3/2}}{3} + D$$

(J) $\int 5e^{(x)}dx = 5\int e^{(x)}dx = 5\left(e^{(x)}+C\right) = 5e^{(x)}+5C$

constant multiple rule

$= 5e^{(x)} + D$.

(*K*) $\int 5dx$

Rewrite the integrand $5 = 5\cdot 1 = 5\cdot x^0$

power function form

$\int 5dx = \int 5\cdot x^0 dx = 5\int x^0 dx = 5\left(\frac{x^{0+1}}{0+1} + c\right)$

$n=0$

constant multiple rule

power rule with $n=0$

$= 5\left(\frac{x^1}{1} + c\right) = 5(x + c) = 5x + 5c$

$= \boxed{5x + C}$

The integral equation $\int 5dx = 5x + C$

corresponds to the derivative equation $\frac{d}{dx}(5x + c) = 5$.

(**L**) $\int 5\ln(x)\,dx$

$$\int 5\ln(x)\,dx = 5\int \ln(x)\,dx = 5\left(x\ln(x) - x + C\right)$$

constant multiple rule

ln(x) rule

$$= 5x\ln(x) - 5x + 5C$$

$$\int 5\ln(x)\,dx = 5x\ln(x) - 5x + D$$