Subject for this video:

Rewrite the Integrand First!

Reading:

- **General:** Section 5.1 Antiderivatives and Indefinite Integrals
- **More Specifically:** Pages 326 - 330, Examples 2,3

Homework: H71: Rewrite the Integrand First! (5.1#43,45,47,49,51,53)

Recall the definition of *antiderivative* from previous videos.

Definition of Antiderivative

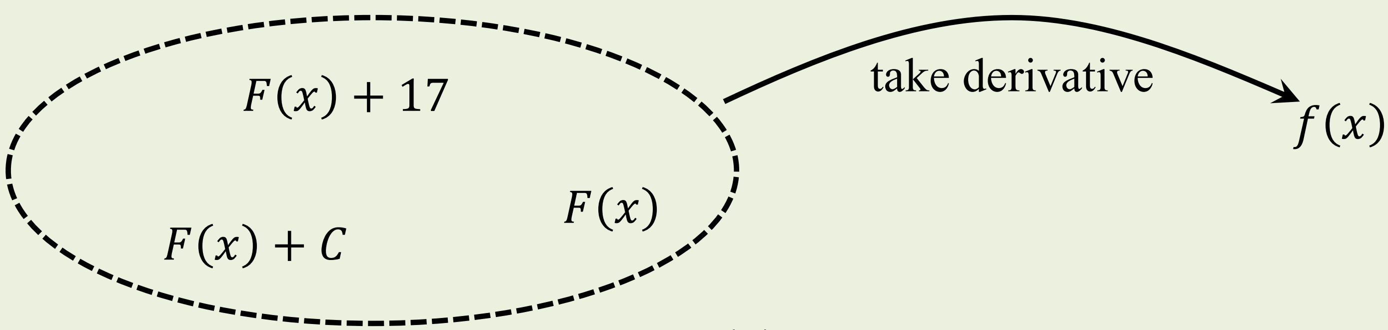
Words: *F is an antiderivative of f.*

Meaning: f is the derivative of F. That is, $f = F'$.

Arrow diagram: $F \xrightarrow{\text{take derivative}} f$

Theorem about the Collection of Antiderivatives of a Function

If a function $F(x)$ is an antiderivative of a function $f(x)$, then any function of the form $F(x) + C$ where C is a real number constant, will also be an antiderivative of $f(x)$. Furthermore, these are *all* the antiderivatives of $f(x)$. That is, if $G(x)$ is also an antiderivative of $f(x)$, then it must be that $G(x) = F(x) + C$ where C is a real number constant.



the collection of all antiderivatives of $f(x)$

Recall the definition of *particular antiderivative* and *general antiderivative*

When a choice of an *actual number* for C is made, the resulting function is called a ***particular antiderivative*** of $f(x)$. That is,

- The function $F(x) = \frac{x^3}{3}$ is a *particular antiderivative* of $f(x) = \frac{x^3}{3}$.
- The function $G(x) = \frac{x^3}{3} + 17$ is a *particular antiderivative* of $f(x)$.

But if C has not been chosen, then the *function form*

$$y = \frac{x^3}{3} + C$$

is called ***the general antiderivative*** of $f(x)$.

Recall the definition of *indefinite integral*

Definition of Indefinite Integral

symbol: $\int f(x)dx$

spoken: the ***indefinite integral*** of $f(x)$

meaning: the ***general antiderivative*** of $f(x)$

Remark: We know that, given one function $F(x)$ that is known to be an antiderivative of $f(x)$, we can get all other antiderivatives by adding constants to $F(x)$. The *general antiderivative* of $f(x)$ is denoted by writing $F(x) + C$, where C is a constant that can be any real number. That is,

$$\text{If } F'(x) = f(x) \text{ then } \int f(x)dx = F(x) + C$$

Additional Terminology:

- The function $f(x)$ inside the integral symbol is called the ***integrand***.
- The unknown constant C is called the ***constant of integration***.

Recall that the *indefinite integral* can be illustrated by the diagrams below.

Suppose:

$$\begin{tikzcd} F(x) \arrow[rr, bend left, "\text{take derivative}"] & & f(x) \end{tikzcd}$$

an antiderivative of $f(x)$

Then:

$$\begin{tikzcd} \int f(x)dx = F(x) + C \arrow[rr, bend left, "\text{take derivative}"] & & f(x) \arrow[ll, bend left, "\text{find indefinite integral}"] \end{tikzcd}$$

general antiderivative of $f(x)$

And finally, recall the *Indefinite Integral Rules* that we discussed in the previous video, presented here along with two new rules.

The *power rule*:	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$
The $\frac{1}{x}$ *rule*:	$\int \frac{1}{x} dx = \ln(\lvert x \rvert) + C$ for all $x \neq 0$
The $e^{(x)}$ *rule*:	$\int e^{(x)} dx = e^{(x)} + C$
The $\ln(x)$ *rule*:	$\int \ln(x)\, dx = x\ln(x) - x + C$
The *constant multiple rule*:	$\int af(x)dx = a\int f(x)dx$
The *sum rule*:	$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$
The *sum and constant multiple rule*:	$\int af(x) \pm bg(x)dx = a\int f(x)dx \pm b\int g(x)dx$

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[Example 1] Find indefinite integrals $(A) - (G)$.

(If the integrand needs to be rewritten, do that *first*, *before* doing the integral!)

(A) $\int \underbrace{17x(3-x)} dx$

rewrite the integrand $f(x) = 17x(3-x) = 51x - 17x^2$

Power function form

Now integrate

$$\int f(x)\,dx = \int 51x - 17x^2\,dx \underset{\uparrow}{=} 51\int \overset{n=1}{x}\,dx - 17\int \overset{n=2}{x^2}\,dx$$

Sum and Constant multiple rule

use power rule

$$= 51\left(\frac{x^{1+1}}{1+1}\right) - 17\left(\frac{x^{2+1}}{2+1}\right) + C$$

$$= 51\frac{x^2}{2} - \frac{17x^3}{3} + C$$

$$= \frac{51x^2}{2} - \frac{17x^3}{3} + C$$

Conclude that $\int 17x(3-x)\,dx = \frac{51x^2}{2} - \frac{17x^3}{3} + C$

(**B**) $\int \frac{15}{\sqrt{u}}\,du$

first rewrite the integrand $\frac{15}{\sqrt{u}} = \frac{15}{u^{1/2}} = 15u^{-1/2}$ power function form

$$\int \frac{15}{\sqrt{u}}\,du = \int \underline{15}\,u^{-1/2}\,du = \underline{15}\int u^{-1/2}\,du \qquad n = -\frac{1}{2}$$

constant multiple rule

power rule with $n = -\frac{1}{2}$

$$= 15\left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C\right) = 15\left(\frac{u^{1/2}}{1/2} + C\right) =$$

$$= 15\left(2u^{1/2} + C\right) = 30u^{1/2} + 15C$$

$$= 30\sqrt{u} + D$$

Conclude $\int \frac{15}{\sqrt{u}}\,du = 30\sqrt{u} + D$

(C) $\int \frac{1}{5x^{17}}\,\cancel{dx}\,dx$

rewrite the integrand $\frac{1}{5x^{17}} = \frac{1}{5}\cdot\frac{1}{x^{17}} = \frac{1}{5}\cdot x^{-17}$

power function form

$n = -17$

$$\int \frac{1}{5x^{17}}\,dx = \int \left(\frac{1}{5}\right) x^{-17}\,dx = \left(\frac{1}{5}\right)\int x^{-17}\,dx$$

constant multiple rule

power rule with $n=-17$

$$= \left(\frac{1}{5}\right)\left(\frac{x^{-17+1}}{-17+1} + C\right) = \left(\frac{1}{5}\right)\left(\frac{x^{-16}}{-16} + C\right) =$$

$$= \left(\frac{1}{5}\right)\left(-\frac{1}{16x^{16}} + C\right) = -\frac{1}{(5)16x^{16}} + \left(\frac{1}{5}\right)C$$

$$= -\frac{1}{80x^{16}} + D$$

Conclude $\int \frac{1}{5x^{16}}\,dx = -\frac{1}{80x^{16}} + D$

(**D**) $\int 13x^5 - \frac{11}{x^5}\,dx$

Rewrite the integrand $13x^5 - \frac{11}{x^5} = 13x^5 - 11x^{-5}$

Power function form

$$\int 13x^5 - \frac{11}{x^5}\,dx = \int 13x^5 - 11x^{-5}\,dx = 13\int x^5\,dx - 11\int x^{-5}\,dx$$

Sum and constant multiple rule

now use power rule with $n=5$ and $n=-5$

$$= 13\left(\frac{x^{5+1}}{5+1}\right) - 11\left(\frac{x^{-5+1}}{-5+1}\right) + C$$

$$= \frac{13x^6}{6} - \frac{11x^{-4}}{-4} + C$$

$$= \frac{13x^6}{6} + \frac{11}{4x^4} + C$$ positive exponent form

Conclude that $\int 13x^5 - \frac{11}{x^5}\,dx = \frac{13x^6}{6} + \frac{11}{4x^4} + C$

(E) $\int \frac{2}{\sqrt[3]{x}} - 5\sqrt[3]{x}\,dx$

rewrite integrand $f(x) = \frac{2}{\sqrt[3]{x}} - 5\sqrt[3]{x} = 2\cdot\frac{1}{x^{1/3}} - 5x^{1/3} = 2x^{-1/3} - 5x^{1/3}$

$\int f(x)\,dx = \int 2x^{-1/3} - 5x^{1/3}\,dx = 2\int x^{-1/3}\,dx - 5\int x^{1/3}\,dx$

Sum & constant multiple rule

Power rule

$= 2\left(\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right) - 5\left(\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right) + C = 2\frac{x^{\frac{2}{3}}}{\frac{2}{3}} - 5\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C$

$= 2\cdot\left(\frac{3}{2}\right)x^{\frac{2}{3}} - 5\cdot\left(\frac{3}{4}\right)x^{\frac{4}{3}} + C = 3x^{\frac{2}{3}} - \frac{15x^{4/3}}{4} + C$

Conclude $\int \frac{2}{\sqrt[3]{x}} - 5\sqrt[3]{x}\,dx = 3x^{\frac{2}{3}} - \frac{15x^{4/3}}{4} + C$

$(\boldsymbol{F})\quad \int \frac{1-x^2}{3x}\,dx$

rewrite integrand $f(x) = \dfrac{1-x^2}{3x} = \dfrac{1}{3x} - \dfrac{x^2}{3x} = \dfrac{1}{3}\cdot\dfrac{1}{x} - \dfrac{1}{3}\cdot x$

$$\int f(x)\,dx = \int \frac{1}{3}\cdot\frac{1}{x} - \frac{1}{3}x\,dx = \frac{1}{3}\int\frac{1}{x}\,dx - \frac{1}{3}\int \overset{n=1}{x}\,dx =$$

use $\frac{1}{x}$ rule

use power rule with $n=1$

$$= \frac{1}{3}\ln(|x|) - \frac{1}{3}\left(\frac{x^{1+1}}{1+1}\right) + C$$

$$= \frac{1}{3}\ln(|x|) - \frac{1}{3}\frac{x^2}{2} + C$$

$$= \frac{1}{3}\ln(|x|) - \frac{x^2}{6} + C$$

Conclude $\displaystyle\int \frac{1-x^2}{3x}\,dx = \frac{1}{3}\ln(|x|) - \frac{x^2}{6} + C$

(G) $\int 7e^x + 5dx$

rewrite the integrand $f(x) = 7e^{(x)} + 5 = 7e^{(x)} + \underbrace{5 \cdot 1}_{\text{power function form.}}$

$\int f(x)\,dx = \int \underline{7}e^{(x)} + \underline{5} \cdot 1\, dx = \underline{7}\int e^{(x)} dx + \underline{5}\int 1\, dx$

$1 = x^0$

Sum & constant multiple rule

$e^{(x)}$ rule

Power rule with $n=0$

$= 7\left(e^{(x)}\right) + 5\left(\dfrac{x^{0+1}}{0+1}\right) + C$

$= 7e^{(x)} + \dfrac{5x^1}{1} + C$

$= 7e^{(x)} + 5x + C$

Conclude $\int 7e^{(x)} + 5dx = 7e^{(x)} + 5x + C$

[Example 2] Valid and Invalid Solutions to an Indefinite Integral Problem

One of these five solutions is correct. Each of the other solutions has at least one invalid step.

The equal signs are all numbered. Discuss the validity or invalidity of each equal sign.

Conclude by saying which solution is the correct one.

$n = -5$

$$(a)\int \frac{2}{7x^5}dx \;\cancel{\underset{(1)}{=}}\; \int 2(7x^{-5})dx \underset{(2)}{=} \cancel{2}\left(\frac{7x^{-4}}{-\cancel{4}}\right) + C \underset{(3)}{=} -\frac{7x^{-4}}{2} + C \underset{(4)}{=} -\frac{7}{2x^4} + C$$

invalid

can't do that   power rule   cancel 2

$$(b)\int \frac{2}{7x^5}dx \underset{(1)}{=} \int \frac{2x^{-5}}{7}dx \underset{(2)}{=} \frac{2x^{-4}}{7(-4)} + C \;\cancel{\underset{(3)}{=}}\; \frac{1}{7(-4)(2x^4)} + C \underset{(4)}{=} -\frac{1}{56x^4} + C$$

invalid

power rule   can't do that

$$(c)\int \frac{2}{7x^5}dx \underset{(1)}{=} \int \frac{2x^{-5}}{7}dx \underset{(2)}{=} \frac{2x^{-4}}{7(-4)} + C \underset{(3)}{=} -\frac{x^{-4}}{7(2)} + C \underset{(4)}{=} -\frac{1}{14x^4} + C$$

valid

power rule

can't do that

invalid

$$(d)\int \frac{2}{7x^5}dx \;\cancel{\underset{(1)}{=}}\; \int 2(7x^{-5})dx \underset{(2)}{=} 2\left(\frac{7x^{-4}}{-4}\right) + C \underset{(3)}{=} -\frac{7x^{-4}}{2} + C \;\cancel{\underset{(4)}{=}}\; -\frac{1}{2(7x^4)} + C \underset{(5)}{=} -\frac{1}{14x^4} + C$$

can't do that   power rule

incorrect

$$(e)\int \frac{2}{7x^5}dx \underset{(1)}{=} \int \frac{2x^{-5}}{7}dx \;\cancel{\underset{(2)}{=}}\; \frac{2x^{-6}}{7(-6)} + C \underset{(3)}{=} -\frac{x^{-6}}{7(3)} + C \underset{(4)}{=} -\frac{1}{21x^6} + C$$

invalid

Solution: We must consider the validity of each equal sign.

Steps in Solution $(\boldsymbol{a})$

1) **Invalid.** Moving the x^5 to the numerator and changing the sign of its exponent was valid, but it is not valid to just move the 7 from the denominator to the numerator.
2) Valid: Used the Power Rule for Indefinite Integrals, with $n = -4$.
3) Valid: Cancelled a 2 and moved the minus sign out front.
4) Valid: Moved the x^{-4} to the denominator and changed the sign of its exponent.

Conclude that Solution $(\boldsymbol{a})$ is invalid because it contains an invalid step.

Steps in Solution $(\boldsymbol{b})$

1) Valid. Moved the x^5 to the numerator and changed the sign of its exponent.
2) Valid: Used the Power Rule for Indefinite Integrals, with $n = -4$.
3) **Invalid.** Moving the x^{-4} to the numerator and changing the sign of its exponent was valid, but it is not valid to just move the 2 from the numerator to the denominator.
4) Valid: Multiplied the constants and moved the minus sign out front.

Conclude that Solution $(\boldsymbol{b})$ is invalid because it contains an invalid step.

Steps in Solution $(\boldsymbol{c})$

1) Valid. Moved the x^5 to the numerator and changed the sign of its exponent.
2) Valid: Used the Power Rule for Indefinite Integrals, with $n = -4$.
3) Valid: Cancelled a 2 and moved the minus sign out front.
4) Valid: Moved the x^{-4} to the denominator, changed the sign of its exponent, and multiplied the constants.

Conclude that Solution $(\boldsymbol{c})$ is valid because all of it steps are valid.

Steps in Solution $(\boldsymbol{d})$

1) **Invalid.** Moving the x^5 to the numerator and changing the sign of its exponent was valid, but it is not valid to just move the 7 from the denominator to the numerator.
2) Valid: Used the Power Rule for Indefinite Integrals, with $n = -4$.
3) Valid: Cancelled a 2 and moved the minus sign out front.
4) **Invalid.** Moving the x^{-4} to the numerator and changing the sign of its exponent was valid, but it is not valid to just move the 7 from the numerator to the denominator.
5) Valid: Multiplied the constant.

Conclude that Solution $(\boldsymbol{d})$ is invalid because it contains two invalid steps.

Steps in Solution (e)

1) Valid. Moved the x^5 to the numerator and changed the sign of its exponent.
2) **Invalid:** Used $n - 1$ instead of $n + 1$ in the Power Rule for Indefinite Integrals.
3) Valid: Cancelled a 2 and moved the minus sign out front.
4) Valid: Moved the x^{-6} to the denominator, changed the sign of its exponent, and multiplied the constants.

Conclude that Solution (e) is invalid because it contains an invalid step.

Conclusion: Solution (c) is a valid solution because all its steps are invalid. All the other solutions are invalid, because each contains at least one invalid step.

Remark: Notice that solution (d) is invalid, even though it led to the correct expression in the end. In fact, solution (d) is the worst solution of the five, because it contains **two** invalid steps!

End of [Example 2]

End of Video