Subject for this video:

Harder Substitution Integrals

Reading:

- **General:** Section 5.2 Integration by Substitution
- **More Specifically:** Page 333 - 343, Examples 4,5,6

Homework: H75: Harder Substitution Integrals: Leftover Multiplicative Constant (5.2#23,27,29,31,33,41,65,67)

The *power rule*:	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$
The $\frac{1}{x}$ *rule*:	$\int \frac{1}{x} dx = \ln(\lvert x \rvert) + C$ for all $x \neq 0$
The $e^{(x)}$ *rule*:	$\int e^{(x)} dx = e^{(x)} + C$
The $\ln(x)$ *rule*:	$\int \ln(x)\, dx = x\ln(x) - x + C$
The *constant multiple rule*:	$\int af(x)dx = a\int f(x)dx$
The *sum rule*:	$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$
The *sum and constant multiple rule*:	$\int af(x) \pm bg(x)dx = a\int f(x)dx \pm b\int g(x)dx$

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The Substitution Method for finding the *indefinite integral*

$$F(x) = \int f(x)\,dx$$

where the integrand $f(x)$ involves a *nested function*.

Step 1 Identify the inner function and call it $\boldsymbol{u}$. Write the equation $inner(x) = u$ to introduce the single letter u to represent the inner function. Circle the equation.

Step 2 Build the equation $\boldsymbol{dx = \frac{1}{u'}du}$. To do this, first find u', then use it to build equation $dx = \frac{1}{u'}du$. Circle the equation.

Step 3 Substitute, Cancel, Simplify. In steps (1) and (2) you have two circled equations. Substitute these into the integrand of your indefinite integral. Cancel as much as possible and simplify by using the *Constant Multiple Rule*. The result should be a new basic integral involving just the variable u. (See **Remarks about Step 3** on the next page.)

Step 4 Integrate. Find the new indefinite integral by using the indefinite integral rules. The result should be a function involving just the variable u (with constant of integration $+C$).

Step 5 Substitute Back. Substitute $u = inner(x)$ into your function from Step (4) The result will be a new function of just the variable x. (Be sure to include the constant of integration $+C$ in your result.) This is the $F(x)$ that we seek.

Remarks about Step 3

Note that the result of **Step 3** should be a new indefinite integral with an integrand that is a function involving the variable u. There are three important things to check at the end of **Step 3**:

- There should be no x in the new indefinite integral. It should involve only u.
- The new indefinite integral should *not* involve a *nested function*, and it should be a *basic integral* that can be integrated using our indefinite integral rules.
- If the above two items are not satisfied, then either you made a mistake, or the original integral might be one for which the Substitution Method cannot be used.

[Example 1] Find the indefinite integral.

$$\int e^{(kx)}\,dx$$

Step 1 identify inner function $kx = u$

Step 2 Build the equation $dx = \frac{1}{u'}du$

$u = kx$

$u' = \frac{du}{dx} = \frac{d}{dx}kx = k$

$dx = \frac{1}{k}du$

Step 3 Substitute, Cancel, Simplify

$$\int e^{(kx)}dx \underset{\uparrow}{=} \int e^{(u)}\frac{1}{k}du \underset{\uparrow}{=} \frac{1}{k}\int e^{(u)}du$$

Substitute

no cancelling to do, but there is a leftover multiplicative constant

simplify using constant multiple rule

Step 4 Integrate $\frac{1}{k}\int e^{(u)}du = \frac{1}{k}\left(e^{(u)} + C\right) = \left(\frac{1}{k}\right)e^{(u)} + \left(\frac{1}{k}\right)\cdot C = \left(\frac{1}{k}\right)e^{(u)} + D$

Step 5 Substitute Back

$$\int e^{(kx)}dx = \left(\frac{1}{k}\right)e^{(kx)} + D$$

The example just completed is important enough to be included on our list of integral rules.

The $\boldsymbol{e^{(x)}}$ *rule for derivatives*:	$\dfrac{d}{dx}e^{(x)} = e^{(x)}$
The $\boldsymbol{e^{(x)}}$ *rule for integrals*:	$\displaystyle\int e^{(x)}dx = e^{(x)} + C$
The $\boldsymbol{e^{(kx)}}$ *rule for derivatives*:	$\dfrac{d}{dx}e^{(kx)} = ke^{(kx)}$
The $\boldsymbol{e^{(kx)}}$ *rule for integrals*:	$\displaystyle\int e^{(kx)}dx = \left(\frac{1}{k}\right)e^{(kx)} + C$

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[Example 2](similar to 5.2#23) Find the indefinite integral.

$$\int (4x-7)^{-5}dx$$

$4x-7=u$

Step1 Identify Inner Function

Step2 Build Equation $dx = \frac{1}{u'}du$

$u = 4x-7$

$u' = \frac{du}{dx} = \frac{d}{dx}\,4x-7 = 4$

$\boxed{dx = \frac{1}{4}du}$

Step3 Substitute, Cancel, Simplify

$$\int (4x-7)^{-5}dx = \int (u)^{-5}\frac{1}{4}du = \frac{1}{4}\int u^{-5}du$$

Substitute

no cancelling to be done but there is a leftover multiplicative constant

use constant multiple rule

Step4 Integrate

$$\frac{1}{4}\int u^{-5}du = \frac{1}{4}\left(\frac{u^{-5+1}}{-5+1}+C\right) = \frac{1}{4}\left(\frac{u^{-4}}{-4}+C\right) =$$

$$= -\left(\frac{1}{16}\right)u^{-4} + \left(\frac{1}{4}\right)C = -\frac{1}{16u^4} + D$$

Step5 Substitute Back

$$\boxed{\int (4x-7)^{-5}dx = -\frac{1}{16(4x-7)^4} + D}$$

[Example 3](similar to 5.2#27) Find the indefinite integral.

$$\int 3xe^{(x^2)}dx$$

Step1 Identify the Inner Function

$$x^2 = u$$

Step2 Build the Equation $dx = \frac{1}{u'}du$

$$u = x^2$$

$$\frac{du}{dx} = \frac{d}{dx}x^2 = 2x$$

$$dx = \frac{1}{2x}du$$

Step3 Substitute, Cancel, Simplify

$$\int 3xe^{(x^2)}dx \underset{\text{Substitute}}{=} \int 3xe^{(u)}\frac{1}{2x}du \underset{\text{Cancel}}{=} \int 3e^{(u)}\frac{1}{2}du \underset{\text{Simplify using constant multiple Rule}}{=} \frac{3}{2}\int e^{(u)}du$$

Step 4 Integrate $\frac{3}{2}\int e^{(u)}du = \frac{3}{2}\left(e^{(u)} + C\right) = \left(\frac{3}{2}\right)e^{(u)} + \left(\frac{3}{2}\right)C =$

$$= \left(\frac{3}{2}\right)e^{(u)} + D$$

Step 5 Substitute Back

$$\int 3xe^{(x^2)}dx = \frac{3}{2}e^{(x^2)} + D$$

[Example 4](similar to 5.2#29) Find the indefinite integral.

$$\int \frac{2}{3x-5}\,dx$$

Step 1 identify the inner function $3x-5=u$

Step 2 Build the equation $dx=\frac{1}{u'}du$

$u=3x-5$

$u'=\frac{du}{dx}=\frac{d}{dx}(3x-5)=3$

$dx=\frac{1}{3}du$

Step 3 Substitute, Cancel, Simplify

$$\int \frac{2}{3x-5}dx \overset{\uparrow}{=} \int \frac{2}{u}\,\frac{1}{3}du \overset{\uparrow}{=} \frac{2}{3}\int \frac{1}{u}du$$

substitute

no cancelling to be done

simplify using the constant multiple rule

Step 4 Integrate $\frac{2}{3}\int \frac{1}{u}du = \frac{2}{3}\left(\ln(|u|)+C\right) =$

$=\left(\frac{2}{3}\right)\ln(|u|)+\left(\frac{2}{3}\right)C=\left(\frac{2}{3}\right)\ln(|u|)+D$

Step 5 Substitute back

$$\int \frac{2}{3x-5}dx = \frac{2}{3}\ln(|3x-5|)+D$$

[Example 5](similar to 5.2#31) Find the indefinite integral.

$$\int e^{(13-17t)}dt$$

Step 1 Identify the inner function $13-17t = u$

Step 2 Build the equation $dx = \frac{1}{u'}du$

$u = 13-17t$

$u' = \frac{du}{dt} = \frac{d}{dt}(13-17t) = -17$

$dt = \left(-\frac{1}{17}\right)du$

Step 3 Substitute, Cancel, Simplify

$$\int e^{(13-17t)}dt = \int e^{(u)}\left(-\frac{1}{17}\right)du = \left(-\frac{1}{17}\right)\int e^{(u)}du$$

Substitute

no cancelling to be done

Simplify using Constant multiple rule

Step 4 Integrate

$$\left(-\frac{1}{17}\right)\int e^{(u)}du = \left(-\frac{1}{17}\right)\left(e^{(u)}+C\right) = \left(-\frac{1}{17}\right)e^{(u)} + \left(-\frac{1}{17}\right)C =$$

$$= -\left(\frac{1}{17}\right)e^{(u)} + D$$

Step 5 Substitute Back

$$\int e^{(13-17t)}dt = -\left(\frac{1}{17}\right)e^{(13-17t)} + D$$

[Example 6](similar to 5.2#33) Find the indefinite integral.

$$\int \frac{x}{13-5x^2}\,dx$$

Step1 Identify the Inner Function

$13-5x^2 = u$

Step2 Build the Equation

$dx = \frac{1}{u'}du$ $\quad u = 13-5x^2$

$u' = \frac{d}{dx}(13-5x^2) = -10x$

$dx = -\frac{1}{10x}du$

Step3 Substitute, Cancel, Simplify

$$\int \frac{x}{13-5x^2}dx = \int \frac{x}{u}\left(-\frac{1}{10x}du\right) = \int\left(-\frac{1}{10}\right)\frac{1}{u}du = -\left(\frac{1}{10}\right)\int \frac{1}{u}du$$

Substitute $\quad$ Cancel $\quad$ Simplify

Step4 Integrate

$$-\frac{1}{10}\int\frac{1}{u}du = -\frac{1}{10}\left(\ln(|u|)+C\right) = -\frac{1}{10}\ln(|u|) - \frac{1}{10}C =$$

$$= -\frac{1}{10}\ln(|u|) + D$$

Step5 Substitute back

$$\int \frac{x}{13-5x^2}dx = -\left(\frac{1}{10}\right)\ln(|13-5x^2|) + D$$

[Example 7](similar to 5.2#41) Find the indefinite integral.

$$\int e^{(5x)}\left(2+3e^{(5x)}\right)^7 dx$$

Step1 Identify the Inner Function

$$2+3e^{(5x)} = u$$

Step2 Build the Equation $dx = \frac{1}{u'}du$

$$u = 2+3e^{(5x)}$$

$$u' = \frac{d}{dx}\left(2+3e^{(5x)}\right) = 0+3\cdot\left(5e^{(5x)}\right)$$

$$u' = 15e^{(5x)}$$

$$dx = \frac{1}{15e^{(5x)}}du$$

Step3 Substitute, Cancel, Simplify

$$\int e^{(5x)}\left(2+3e^{(5x)}\right)^7 dx \underset{\text{Substitute}}{=} \int e^{(5x)}(u)^7 \frac{1}{15e^{(5x)}}du \underset{\text{cancel}}{=} \int\left(\frac{1}{15}\right)u^7du \underset{\text{simplify}}{=} \left(\frac{1}{15}\right)\int u^7du$$

Step4 Integrate

$$\left(\frac{1}{15}\right)\int u^7du = \left(\frac{1}{15}\right)\left(\frac{u^{7+1}}{7+1}+C\right) = \left(\frac{1}{15}\right)\frac{u^8}{8}+\left(\frac{1}{15}\right)C =$$

$$= \left(\frac{1}{120}\right)u^8 + D$$

Step5 Substitute Back

$$\boxed{\int e^{(5x)}\left(2+3e^{(5x)}\right)^7 dx = \left(\frac{1}{120}\right)\left(2+3e^{(5x)}\right)^8 + D}$$

[Example 8](similar to 5.2#65) Find the indefinite integral.

$$\int \frac{x^4}{\sqrt{3x^5+7}}\,dx = \int x^4(3x^5+7)^{-1/2}\,dx$$

$$3x^5+7=u$$

Step 1 Identify the Inner Function

Step 2 Build the Equation $\quad dx = \frac{1}{u'}\,du$

$$u = 3x^5+7$$

$$u' = \frac{d}{dx}(3x^5+7) = 15x^4$$

$$dx = \frac{1}{15x^4}\,du$$

Step 3 Substitute, Cancel, Simplify

$$\int x^4(3x^5-7)^{-1/2}\,dx = \int x^4 (u)^{-1/2}\,\frac{1}{15x^4}\,du = \int u^{-\frac{1}{2}}\frac{1}{15}\,du = \frac{1}{15}\int u^{-\frac{1}{2}}\,du$$

Substitute ↑ Cancel ↑ Simplify ↑

Step 4 Integrate

$$\frac{1}{15}\int u^{-\frac{1}{2}}\,du = \left(\frac{1}{15}\right)\left(\frac{u^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}+C\right) = \left(\frac{1}{15}\right)\left(\frac{u^{1/2}}{1/2}+C\right)$$

$$= \left(\frac{1}{15}\right)\left(2u^{1/2}+C\right) = \left(\frac{2}{15}\right)u^{1/2} + \left(\frac{1}{15}\right)C = \left(\frac{2}{15}\right)u^{1/2} + D$$

Step 5 Substitute Back

$$\boxed{\int \frac{x^4}{\sqrt{3x^5+7}}\,dx = \left(\frac{2}{15}\right)\sqrt{3x^5+7} + D}$$

[Example 9](similar to 5.2#41) Find the indefinite integral.

$$\int \frac{(\ln(x))^{13}}{x}\,dx = \int \frac{1}{x}(\ln(x))^{13}\,dx$$

Step 1 Identify the Inner Function

$$\ln(x) = u$$

Step 2 Build the Equation $dx = \frac{1}{u'}du$

$$u = \ln(x)$$

$$u' = \frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$dx = \frac{1}{\frac{1}{x}}du$$

$$dx = x\,du$$

Step 3 Substitute, Cancel, Simplify

$$\int \left(\frac{1}{x}\right)(\ln(x))^{13}\,dx = \int\left(\frac{1}{x}\right)(u)^{13}x\,du = \int u^{13}\,du$$

Substitute

cancel

no further simplifying to be done

Step 4 Integrate

$$\int u^{13}\,du = \frac{u^{13+1}}{13+1} + C = \frac{u^{14}}{14} + C$$

Step 5 Substitute back

$$\int \frac{(\ln(x))^{13}}{x}\,dx = \frac{(\ln(x))^{14}}{14} + C$$

[Example 10] Find the indefinite integral.

$$\int \ln(x^2)\,dx$$

Use Substitution method

Step1 Identify Inner Function $\qquad x^2 = u$

Step2 Build the Equation $\qquad dx = \frac{1}{u'}\,du$

$u = x^2$

$u' = \frac{d}{dx} x^2 = 2x$

$dx = \frac{1}{2x}\,du$

Step3 Substitute, Cancel, Simplify

$$\int \ln(x^2)\,dx = \int \ln(u)\,\frac{1}{2x}\,du$$

Substitute

This integral involves both x and u. We can't go any farther. The Substitution method Does Not Work for this integral!

Notice that the integrand can be rewritten $\ln(x^2) = 2\ln(x)$

So $\int \ln(x^2)\,dx = \int 2\ln(x)\,dx = 2\int \ln(x)\,dx = 2\left(x\ln(x) - x + C\right)$

Constant multiple rule

$= 2x\ln(x) - 2x + 2C$

So $\int \ln(x^2)\,dx = 2x\ln(x) - 2x + D$