Subject for this video:

Approximating Areas with Sums

Reading:

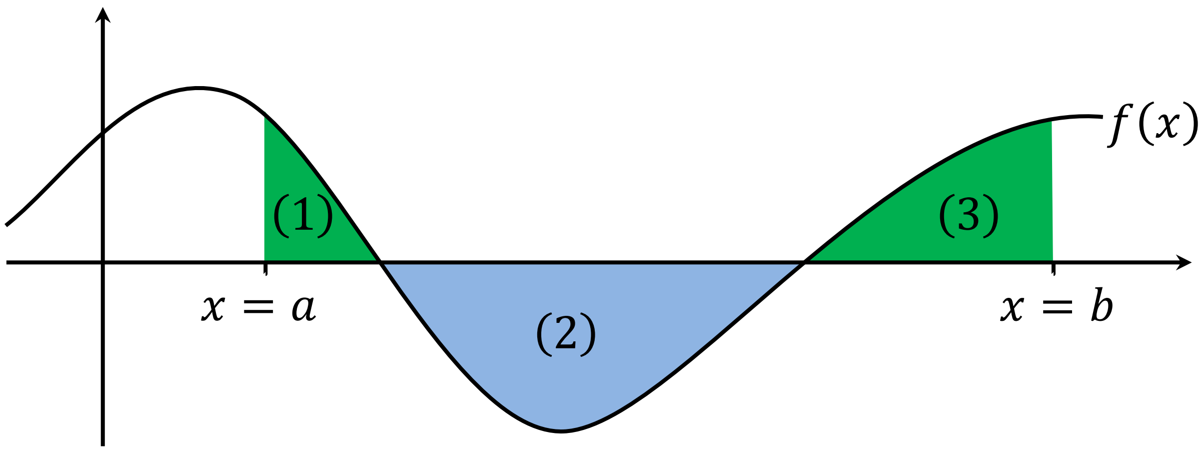
- **General:** Section 5.4 The Definite Integral
- **More Specifically:** Page 358 – top of 364, Examples 1,2

Homework: H76: Approximating Areas with Sums

- Barnett 5.4#7,13,17,19
- Briggs & Cochran 5.1#13,23,30,

Today, we discuss a new idea:

The area between the graph of a function $f(x)$ and the x axis from $x = a$ to $x = b$.



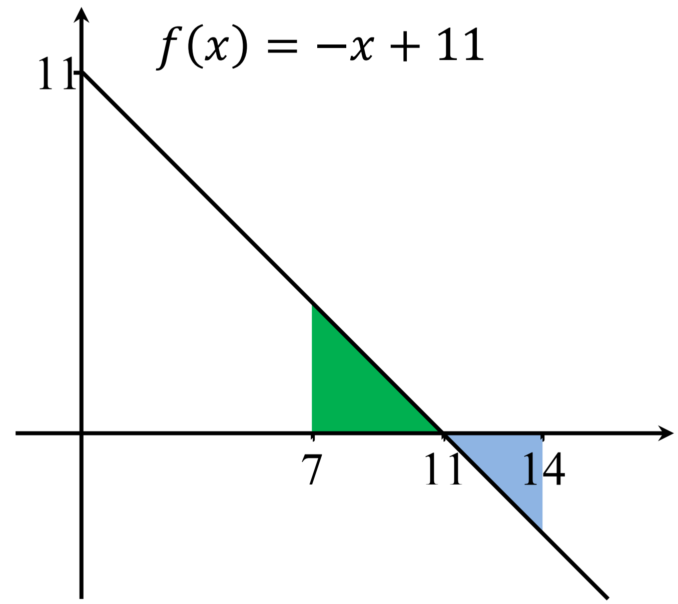
We will be interested in two kinds of area.

- ***Unsigned Area*** (USA) = (1) + (2) + (3)
- ***Signed Area*** (SA) = (1) - (2) + (3) (Regions under the x axis get a negative sign.)

Signed Area may seem like a weird quantity to consider, but mathematically, it is very important. We will discuss it extensively for the rest of the semester. Today, we will consider examples of computing unsigned area and signed area. In simple examples, we can do this using geometry.

[Example 1] Let $f(x) = -x + 11$.

Find the unsigned and signed areas between the graph of $f(x)$ and the x axis from $x = 7$ to $x = 14$ by using geometry



Solution:

The graph of $f(x) = -x + 11$ will be a line with

- slope $m = -1$
- y intercept at $(x, y) = (0,11)$
- x intercept at $(x, y) = (11,0)$

It is drawn at right, with the region between the graph of $f(x)$ and the x axis from $x = 7$ to $x = 14$ shaded.

Green region is a right triangle with base $b = 4$, height $h = 4$ and area $A = \frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot 4 = 8$.

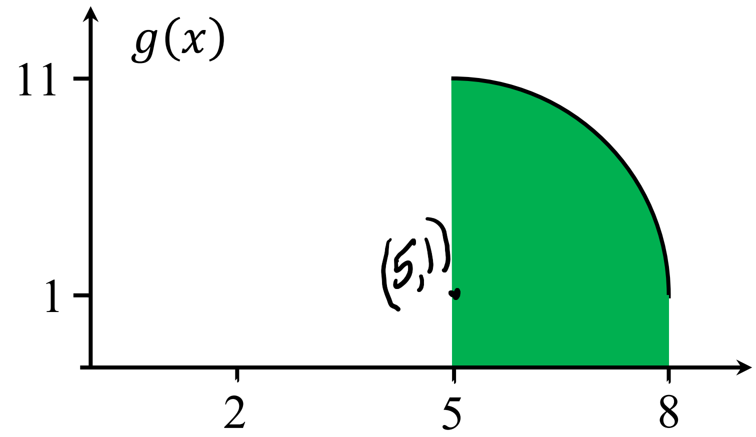
Blue region is a right triangle with base $b = 3$, height $h = 3$, and area $A = \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$.

The unsigned area is $USA = green + blue = 8 + \frac{9}{2} = \frac{25}{2}$.

The signed area is $SA = green - blue = 8 - \frac{9}{2} = \frac{7}{2}$.

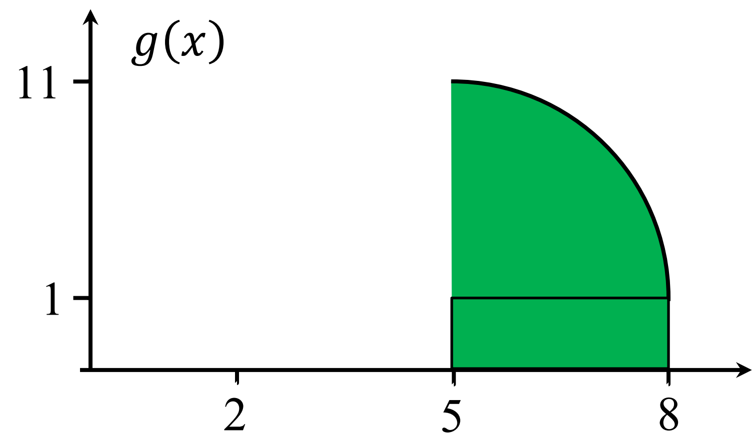
End of [Example 1]

[Example 2] Let $g(x)$ be defined by the graph at right. (The graph of $g(x)$ is part of a circle centered at $(5,1)$.). Find the unsigned and signed areas between the graph of $g(x)$ and the x axis on the interval $[5,8]$ by using geometry. Give an exact answer and a decimal approximation.



Solution:

The top part of the region is a quarter of a circle with radius $r = 3$ and area $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}$.



But the quarter circle region is sitting on a rectangle. The rectangle has width $w = 3$, height $h = 1$, and area $A = w \cdot h = 3$.

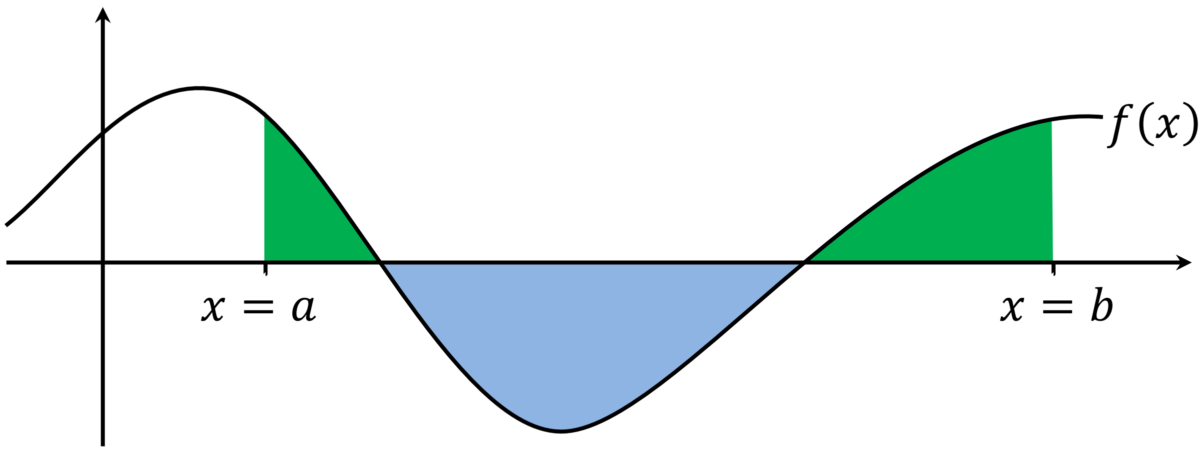
The unsigned area and signed area will be the same because this entire region is above the x axis.

Their value is $USA = SA = green = \underbrace{\frac{9\pi}{4} + 3}_{exact} \approx \underbrace{10.07}_{approx}$.

End of [Example 2]

We have seen that for regions made up of simple geometric shapes, we can get the signed area using geometry.

But what about regions that are not made up of simple geometric shapes? For example, for the function $f(x)$ defined by the curvy graph below, consider the following two questions.



Two Questions about Area

(1) How can we *define* the *signed area* between the graph of a function $f(x)$ and the x axis from $x = a$ to $x = b$ for a general function $f(x)$ whose graph is not made up basic shapes?

(2) How can we find the *value* of that *signed area*?

If you think back to your experience with area in previous courses, you probably never encountered a definition of what area *means*. Rather, you just had various formulas for computing the area of basic shapes, and rules for computing the areas of more complicated regions that area made up of basic shapes.

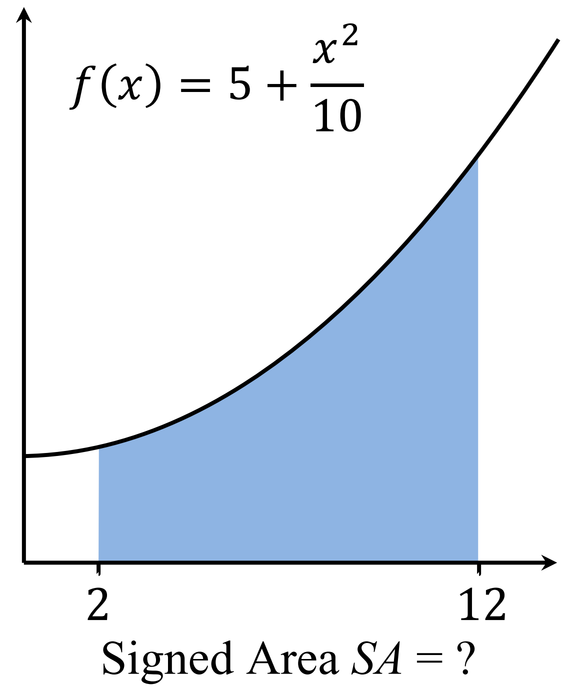
It may seem like we would have to answer *Area Question* (1) before answering *Area Question* (2). But it turns out that we can fumble our way to an answer to Question (1) by first trying to answer Question (2).

Approximating Areas with Rectangles

Our quest to answer the *Area Questions* will begin by considering *Area Question* (2):

How would we find the *value* of the *signed area* between the graph of a function $f(x)$ and the x axis from $x = a$ to $x = b$ for a general function $f(x)$ whose graph is not made up basic shapes?

We will consider doing this for a particular choice of $f(x)$ and interval $[a, b]$.



We will try to find the value of the signed area between the graph of

$$f(x) = 5 + \frac{x^2}{10}$$

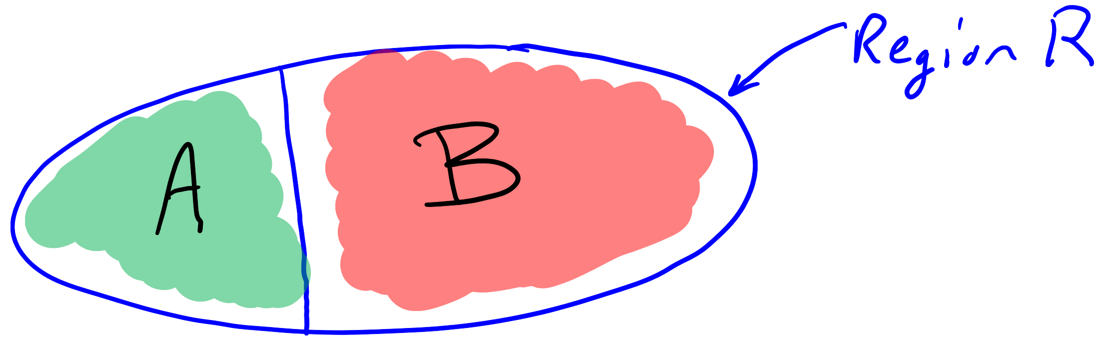
and the x axis from $x = 2$ to $x = 12$.

Remember: we haven't yet defined what *signed area* even means for such a region. (That's what we're trying to figure out.) But we can formulate two requirements that we will ask of the definition of area:

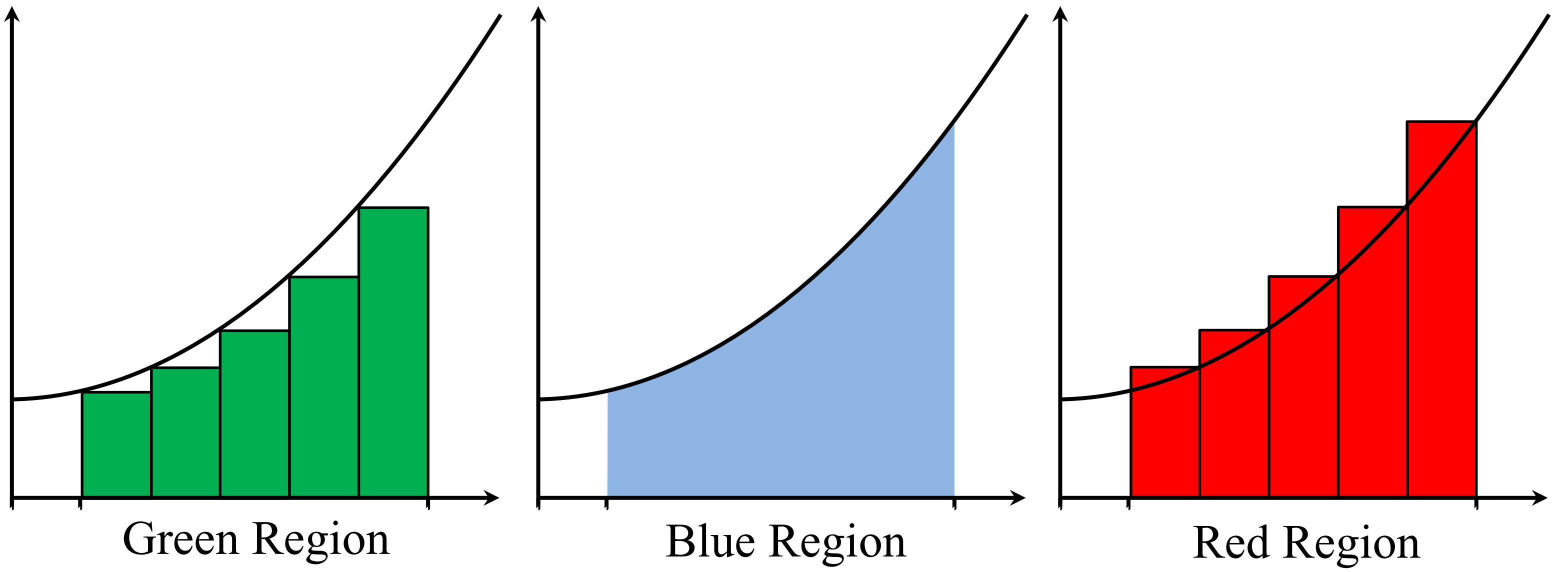
Requirements that we want a Definition of Area to Satisfy:

- ***Positivity***: For any non-empty region R above the x axis, the area of Region R should be a positive number.
- ***Additivity*:** If some region R above the x axis is made up of the disjoint union of regions A and B, then the area of region A plus the area of region B should equal the area of Region R.

As a result of *Positivity* and *Additivity*, it will be true that if some region A above the x asis is entirely contained in some region R above the x axis, then the area of A should be less than the area of R. That idea can be used to give an upper and lower estimate of the signed area of the blue region shown above.



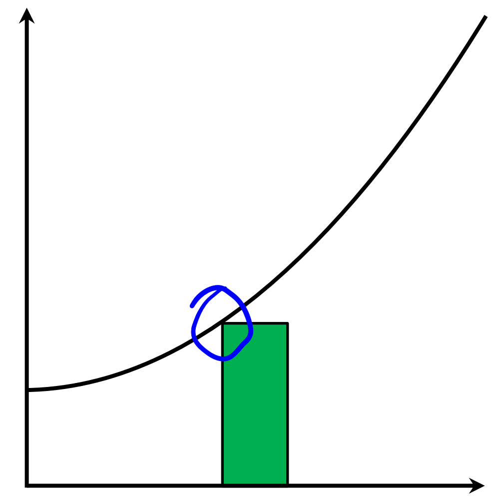
Consider the green, blue, and red shaded regions in the figure below.



We don’t know the value of the signed area SA for the blue region. We have not even defined what signed area means for that region. But the green and red regions are made up of rectanges, so we can find the signed area of those regions. Then, we will have lower and upper bounds for the unknown (and even undefined) area of the blue region. Sort of an “***area sandwich***”. That is the general idea of *approximating areas with rectangles*. We will now set about computing the green and red areas, doing it in a way that the process can be generalized to other examples. The terminology we will be introduced to is that of *left rectangles, right rectangles, left sums, right sums, and more generally, Riemann sums*.

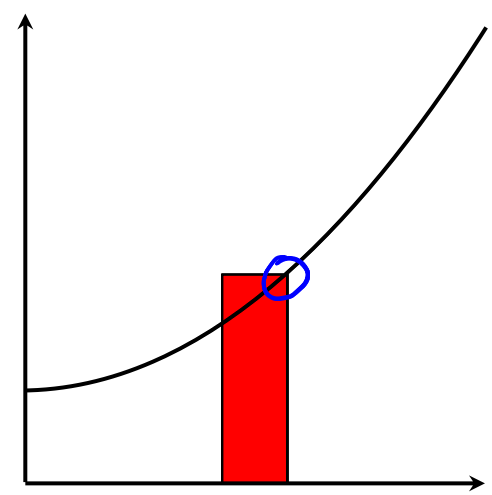
Left and Right Rectangles

For a given function $f(x)$, *Left Rectangles* and *Right Rectangles* will be defined as follows



Definition of *Left Rectangle*

For a given function $f(x)$, a *Left Rectangle* sits on the x axis and is the correct height so that it touches the graph of $f(x)$ at the rectangle's *left* corner.



Definition of *Right Rectangle*

For a given function $f(x)$, a *Right Rectangle* sits on the x axis and is the correct height so that it touches the graph of $f(x)$ at the rectangle's *right* corner.

Remark: These definitions sound simple enough. However, notice that if the graph of $f(x)$ is below the x axis, the rectangles will have to go down to touch the graph. And if the graph of $f(x)$ happens to have an x intercept, then a rectangle may have no height at all if its corner happens to occur at that x value.

Left and Right Sums

A *Left Sum* will be defined to be a number that obtained by adding the areas of a bunch of *Left Rectangles* in a certain way.

Definition of *Left Sum*

Symbol: L_n

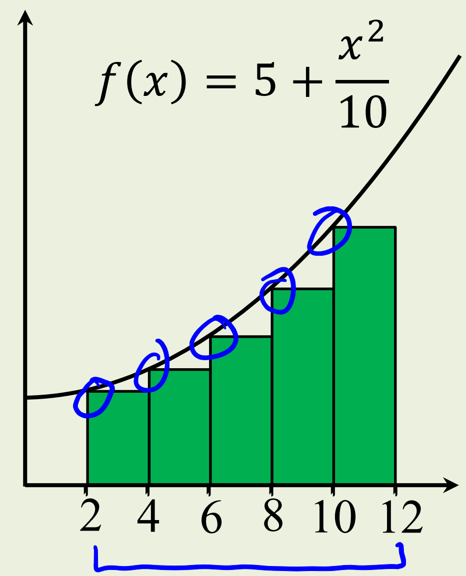
Spoken: The *Left Sum with n rectangles*

Usage: a continuous function $f(x)$ and an interval $[a, b]$ are given

Meaning:

- put n equal-width *Left Rectangles* on the interval $a \leq x \leq b$.
- Add up their area.
- The resulting sum is denoted L_n

Picture: Here is a picture of the 5 rectangles that would be used to compute L_5 for $f(x) = 5 + \frac{x^2}{10}$ on the interval $[2,12]$.



Similarly for the definition of a *Right Sum*.

Definition of *Right Sum*

Symbol: R_n

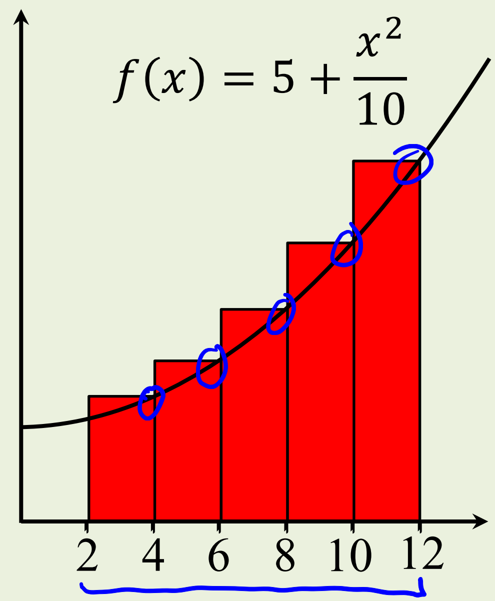
Spoken: The *Right Sum with* n *rectangles*

Usage: a continuous function $f(x)$ and an interval $[a, b]$ are given

Meaning:

- put n equal-width *Right Rectangles* on the interval $a \leq x \leq b$.
- Add up their area.
- The resulting sum is denoted R_n

Picture: Here is a picture of the 5 rectangles that would be used to compute R_5 for $f(x) = 5 + \frac{x^2}{10}$ on the interval $[2,12]$.



Riemann Sums

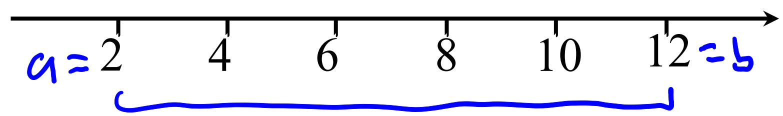
The *Left Sum* and *Right Sum* are two examples of a whole family of sums called *Riemann Sums*, named for German mathematician Bernhard Riemann (1826 – 1866) Riemann was an important figure in the development of the theory of the integral.

Computing Left and Right Riemann Sums

We will compute one example involving a particular function $f(x)$ and then present the general procedure.

[Example 3] Find the *Left Sum* and *Right Sum* with 5 rectangles for $f(x) = 5 + \frac{x^2}{10}$ on the interval $[2,12]$. That is, find the values of L_5 and R_5.

The regions corresponding to L_5 and R_5 each have five rectangles. To compute their areas, we will need their widths and their heights.



Notice that all of the rectangles will have the same width, $w = 2$.

That width is obtained by dividing the width of the whole interval by the number of rectangles:

$$2 = \frac{12-2}{5} = \frac{b-a}{n}$$

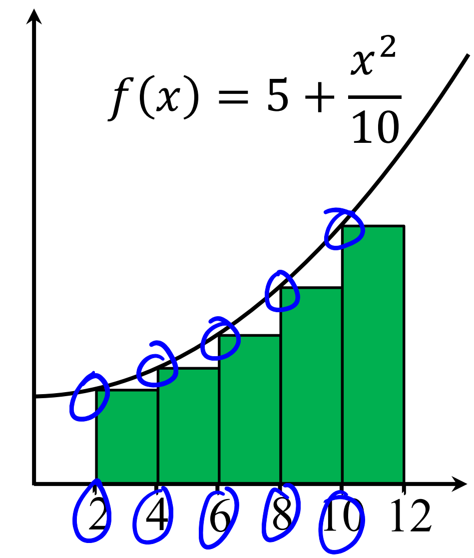
For the sake of generalizing this procedure, it is helpful to name the rectangle width and make note of how it is computed in general.

$$\boxed{rectangle\ width = \Delta x = \frac{b-a}{n}}$$

.

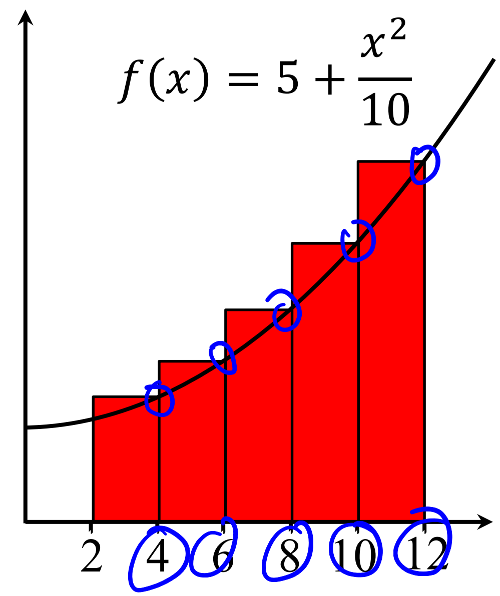
The heights of the five *left* rectangles are the same as the height of the graph of $f(x)$ at the *left* edges of the green rectangles. (Because these are *left* rectangles.)

Those left edges are located at the x values 2,4,6,8,10.



The heights of the five *right* rectangles are the same as the height of the graph of $f(x)$ at the *right* edges of the red rectangles. (Because these are *right* rectangles.)

Those right edges are located at the x values 4,6,8,10,12.



Notice that there are six important x values involved in computing the heights of the five left rectangles and the five right rectangles.

For the sake of generalizing this procedure later, it is helpful to name those six x important values and observe how they are computed.

Computing the six important x values.

$$x_0 = 2 = a$$

$$x_1 = 4 = 2 + 2 = a + \Delta x$$

$$x_2 = 6 = 2 + 2(2) = a + 2\Delta x$$

$$x_3 = 8 = 2 + 3(2) = a + 3\Delta x$$

$$x_4 = 10 = 2 + 4(2) = a + (n - 1)\Delta x$$

$$x_5 = 12 = 2 + 5(2) = a + n\Delta x = b$$

The heights of the five *left* rectangles will be the five y values $f(2), f(4), f(6), f(8), f(10)$, and the heights of the five *right* rectangles will be the five y values $f(4), f(6), f(8), f(10), f(12)$. We see that it will be necessary to compute the y values for all six of the important x values listed.

$$f(2) = 5 + \frac{(2)^2}{10} = 5 + \frac{4}{10} = \frac{54}{10}$$

$f(x) = 5 + \frac{x^2}{10}$

$$f(4) = 5 + \frac{(4)^2}{10} = 5 + \frac{16}{10} = \frac{66}{10}$$

$$f(6) = 5 + \frac{(6)^2}{10} = 5 + \frac{36}{10} = \frac{86}{10}$$

$f(\ ) = 5 + \frac{(\ )^2}{10}$ empty version

$$f(8) = 5 + \frac{(8)^2}{10} = 5 + \frac{64}{10} = \frac{114}{10}$$

$$f(10) = 5 + \frac{(10)^2}{10} = 5 + \frac{100}{10} = \frac{150}{10}$$

$$f(12) = 5 + \frac{(12)^2}{10} = 5 + \frac{144}{10} = \frac{194}{10}$$

For the sake of generalizing this procedure later, notice that those six y values can be denoted.

$$\boxed{\textit{The y vlaues are } f(x_0), f(x_1), f(x_2), f(x_3), f(x_4), \cancel{f(x_6)}}$$

$f(x_5)$

.

Finally, we compute the sums.

The *Left Sum* L_5 will be the sum of the areas of the five green *left* rectangles, with each area being a *height* times a *width*. The five heights are $f(2), f(4), f(6), f(8), f(10)$; the widths are all $\Delta x = 2$.

$$\begin{aligned}
L_5 &= f(2)\cdot \underline{2} + f(4)\cdot \underline{2} + f(6)\cdot \underline{2} + f(8)\cdot \underline{2} + f(10)\cdot \underline{2} \\
&= \big(f(2) + f(4) + f(6) + f(8) + f(10)\big)\cdot \underline{2} \\
&= \left(\frac{54}{10} + \frac{66}{10} + \frac{86}{10} + \frac{114}{10} + \frac{150}{10}\right)\cdot 2 \\
&= \left(\frac{470}{10}\right)\cdot 2 \\
&= 47\cdot 2 \\
&= 94
\end{aligned}$$

We have found a value for the *Left Sum*: $L_5 = 94$.

The *Right Sum* R_5 will be the sum of the areas of the five red *right* rectangles, with *heights* $f(4), f(6), f(8), f(10), f(12)$ and *widths* $\Delta x = 2$.

$$
\begin{aligned}
R_5 &= f(4)\cdot 2 + f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + f(12)\cdot 2 \\
&= \big(f(4) + f(6) + f(8) + f(10) + f(12)\big)\cdot 2 \\
&= \left(\frac{66}{10} + \frac{86}{10} + \frac{114}{10} + \frac{150}{10} + \frac{194}{10}\right)\cdot 2 \\
&= \left(\frac{610}{10}\right)\cdot 2 \\
&= 61\cdot 2 \\
&= 122
\end{aligned}
$$

We have found a value for the *Right Sum*: $R_5 = 122$.

For the sake of generalizing this procedure later, notice the sums turned out to be

$$L_n = \big(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big) \cdot \Delta x$$
$$R_n = \big(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\big) \cdot \Delta x$$

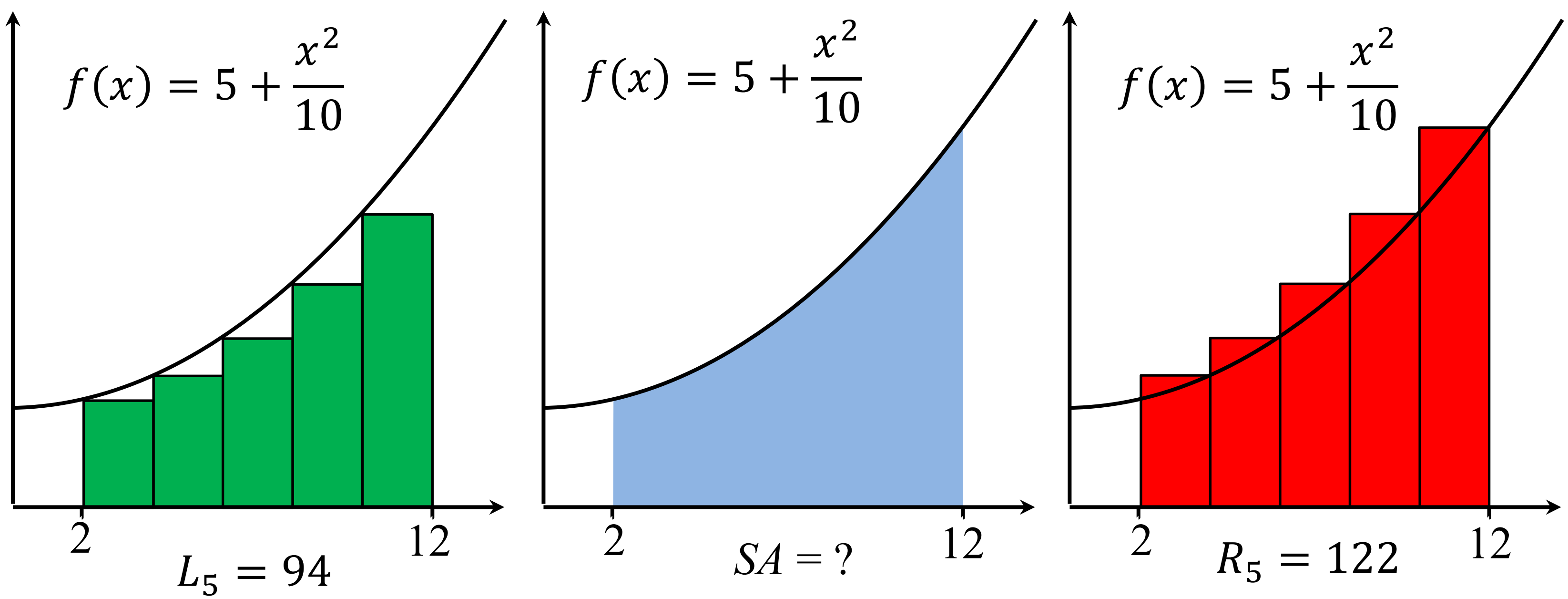
These sums can be abbreviated using **summation notation**. We will not use the summation notation in our course, but it is interesting because it will be the inspiration for a symbol that we will be introduced to later.

$$L_n = \big(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big) \cdot \Delta x = \sum_{k=1}^{n} f(x_{k-1})\Delta x$$

$$R_n = \big(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\big) \cdot \Delta x = \sum_{k=1}^{n} f(x_k)\Delta x$$

In the summation notation, the large symbol is the Greek capital letter *Sigma*.

Discussion of our results: We found $L_5 = 94$ and $R_5 = 122$. Recall our picture illustrating how we were going to find lower and upper bounds for the area of the region between the graph $f(x) = 5 + \frac{x^2}{10}$ and the x axis from $x = 2$ to $x = 12$.



We still don't have a value for the blue region, or a definition of what it even means. But we now have lower and upper bounds for what the area should be.

$$L_5 < SA < R_5$$
$$94 < SA < 122$$

End of [Example 3]

General Procedure

The example we just finished was very long. But along the way, we made note of the important steps in the solution and wrote formulas showing how the calculations for those steps could be generalized. We put those general formulas in four green boxes. The whole procedure can be summarized succinctly into a three-step process by simply gathering up the formulas in those four green boxes and repackaging them in one. It is shown on the next page.

Steps for Computing Left and Right Riemann Sums

Given a continuous function $f(x)$ and an interval $[\mathrm{a}, \mathrm{b}]$,

Step 1: Compute the Rectangle Width $w = \Delta x = \frac{b-a}{n}$

Step 2: Make a list of the x coordinates of the edges of all the rectangles. (Notice, there will be $n+1$ numbers) Find the corresponding y coordinates on the graph of $f(x)$.

x	$y = f(x)$
$x_0 = a$	$y_0 = f(x_0)$
$x_1 = a + \Delta x$	$y_1 = f(x_1)$
$x_2 = a + 2\Delta x$	$y_2 = f(x_2)$
$x_3 = a + 3\Delta x$	$y_3 = f(x_3)$
$\vdots$	$\vdots$
$x_{n-1} = a + (n-1)\Delta x$	$y_{n-1} = f(x_{n-1})$
$x_n = a + n\Delta x = b$	$y_n = f(x_n)$

Step 3: Use the y values on the list and Δx to compute L_n and R_n

$$L_n = \left(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right) \cdot \Delta x$$

$$R_n = \left(f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right) \cdot \Delta x$$

Getting tighter lower and upper bounds

In [Example 3], in our picture showing the region between the graph of $f(x) = 5 + \frac{x^2}{10}$ and the x axis on the interval $[2,12]$ along with the rectangles for L_5 and R_5, notice that there are sizeable spaces between the rectangles and the graph of $f(x)$. That is why our lower and upper bounds of $L_5 = 94$ and $R_5 = 122$ are so far apart. If we used more rectangles, skinnier rectangles, we could eliminate some of that empty space, and our lower and upper bounds would be closer together. We will do that in our next example, and we will let a computer do the hard work.

[Example 4] We will find better lower and upper bounds for the unknown quantity that is the area of the region between the graph of $f(x) = 5 + \frac{x^2}{10}$ and the x axis on the interval $[2,12]$, abbreviated SA for *signed area*.

We will find these better lower and upper bounds by computing the *Left Sum* and *Right Sum* with n rectangles (that is, find the values of L_n and R_n) with higher and higher values of n. To compute the Left and Right Sums, we will use the Riemann Sum Calculator on the GeoGebra web site

Riemann Sum Calculator on GeoGebra Web Site

We will present the results in a table, with a column for L_n and a column for R_n.

Remembering that these will always be lower and upper bounds for the unknown value SA, we will include a column for this unknown quantity. (We will show the results from [Example 3] in the first row of the table.)

Left and Right Riemann Sums for $f(x) = 5 + \frac{x^2}{10}$ on the interval $[2, 12]$.

n	L_n	SA (unknown)	R_n
5	$L_5 = 94$	SA (unknown)	$R_5 = 122$
10	$L_{10} = 100.5$	SA (unknown)	$R_{10} = 114.5$
100	$L_{100} = 106.635$	SA (unknown)	$R_{100} = 108.035$
1000	$L_{1000} = 107.263$	SA (unknown)	$R_{1000} = 107.403$
10000	$L_{10000} = 107.3263$	SA (unknown)	$R_{10000} = 107.3403$
⋮	⋮	⋮	⋮
$n \to \infty$	L_n getting closer to around 107.33	SA (unknown)	R_n getting closer to around 107.33

Based on the table, we can see that unknown (and undefined) value SA of the area of the region between the graph of $f(x) = 5 + \frac{x^2}{10}$ and the x axis on the interval $[2,12]$, must be around

$$SA \approx 107.33$$

End of [Example 4]

Observations:

There are two very important observations to be made about the results of [Example 4].

Observation #1 Higher values of n allow us to give better and better estimates of SA.

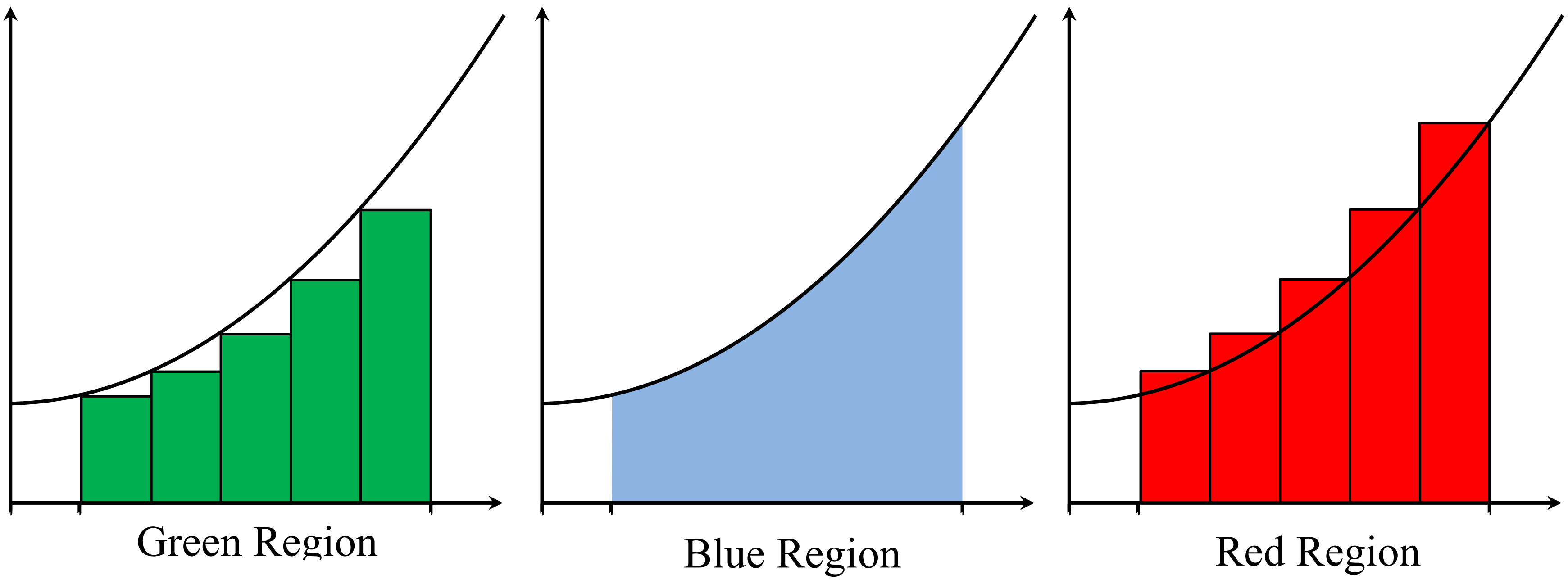
Notice that

$n=5$

- Based on the $n = \cancel{10}$ row of the table, we might guess that $SA \approx 108$.
- Based on the $n = 100$ row of the table, we might guess that $SA \approx 107$.
- Based on the $n = 1000$ row of the table, we might guess that $SA \approx 107.3$.
- Based on the $n = 10{,}000$ row of the table, we might guess that $SA \approx 107.33$.

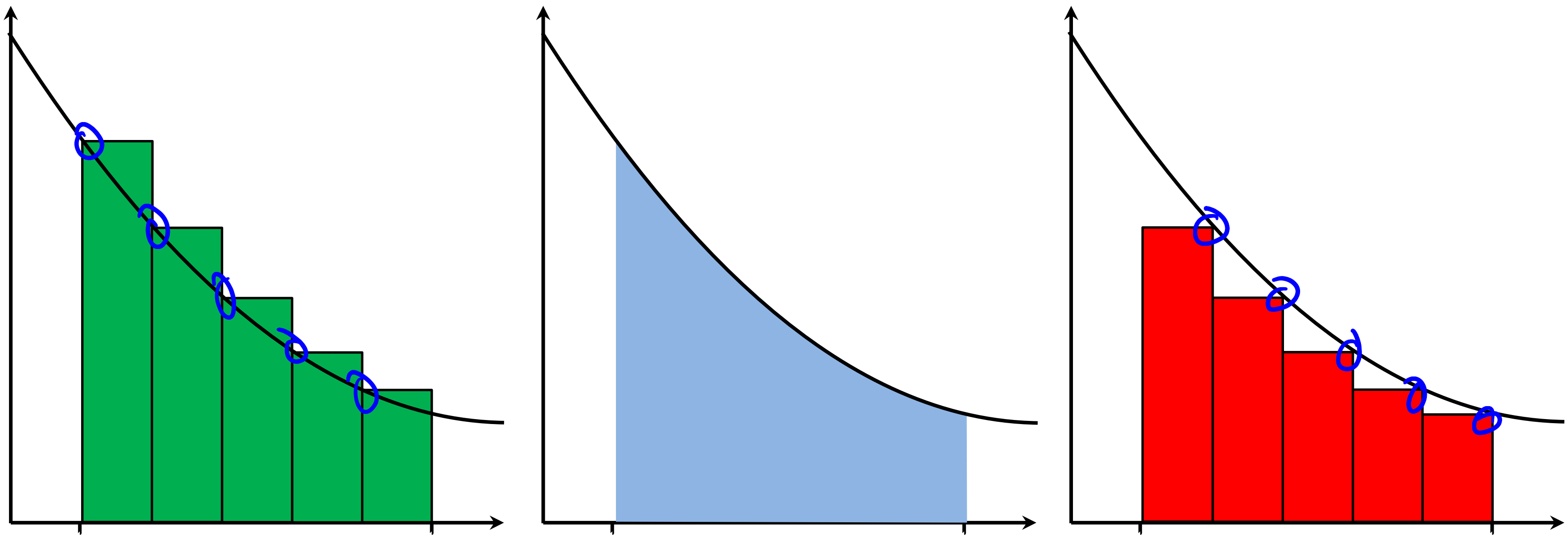
Observation #2 The Area Sandwich won't always occur.

Notice that in every row of the table, $L_n < SA < R_n$. This is because of the "***area sandwich***" that was illustrated in an earlier figure.



Realize that the inequality $L_n < SA < R_n$ occurs because the function $f(x) = 5 + \frac{x^2}{10}$ is increasing on the interval $[2,12]$.

If, instead, we were dealing with a function $g(x)$ that was decreasing on an interval $[a, b]$, we would have the reversed inequalities $L_n > SA > R_n$.



Green Region Blue Region Red Region

And in general, if we were dealing with a function $h(x)$ that was not strictly increasing or decreasing on an interval $[a, b]$, there might not be an area sandwich at all, and so we won't be able to predict the relative sizes of L_n, SA, R_n. This might seem discouraging, because we wanted to use the Riemann sums to find a value of the unknown signed area, and a key part of that was that the Riemann sums provided lower and upper bounds because of the "area sandwich". But luckily, it turns out one does not need the area sandwich:

Big Fact: (This is the sort of thing that the mathematician Riemann proved)

It is a fact from more advanced math that for a function $f(x)$ that is continuous on a closed interval $[a, b]$, the values of L_n and R_n always approach some common number as $n \to \infty$.

That is, if $f(x)$ is continuous on a closed interval $[a, b]$,

then $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} R_n$ both exist and $\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$

Answering both area questions at once.

The **Big Fact** gives us a way to *define* the signed area of the region between the graph of a general curvy function $f(x)$ and the x axis on a closed interval $[a, b]$, and also *compute its value*. We can just *define* the signed area to be the number that is the limit of the Riemann Sums. That is, define

$$Signed\ Area = SA \overset{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

Definition

Notice that this answers *both* of our earlier questions about area:

> **Two Questions about Area**
>
> (1) How can we *define* the *signed area* between the graph of a function $f(x)$ and the x axis from $x = a$ to $x = b$ for a general function $f(x)$ whose graph is not made up basic shapes?
>
> (2) How can we find the *value* of that *signed area*?

Because it answers both questions about area, the definition of signed area is incredibly important.

Because the definition is so important, it gets a name and a symbol.

Definition of the *Definite Integral* and *Signed Area*

Words: The definite integral of $f(x)$ from $x = a$ to $x = b$.

Symbol:

$$\int_{x=a}^{x=b} f(x)dx$$

Alternate Words: The *signed area* of the region between the graph of $f(x)$ and the x axis on the interval $[a, b]$.

Alternate Symbol: SA

Usage: $f(x)$ is continuous on the interval $[a, b]$.

Meaning: the number $\lim_{n\to\infty} L_n$ (which is also the value of $\lim_{n\to\infty} R_n$)

That is,

$$SA = \int_{x=a}^{x=b} f(x)dx \stackrel{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

definition

Remark on Symbol:

We have defined *signed area* to be the value of a limit. We can focus on the limit of the right sum.

$$SA \overset{\text{def}}{=} \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

The symbol chosen for the definite integral is meant to signify that the definite integral is a limit.

- The large *S*-like symbol is meant to meant to evoke kind of a smoothed out greek letter Sigma
- The dx symbol is meant to evoke kind of a smoothed out Δx.

That is,

$$SA = \int_{x=a}^{x=b} f(x)dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Another Big Fact: For graphs made of simple geometric shapes, this definition of signed area gives the same answer as ~~doing~~ finding the signed area using geometric area formulas.

But this is still a little puzzling: We have only seen *one* example of this definition in use. In that example, we have used a *computer* to find L_n and R_n for particular values of n, as n got larger and larger, and we *guessed* at a *rough value* for the limit.

Obvious questions:

Question: Can we do the Riemann Sums L_n and R_n *analytically*, obtaining *general formulas* instead of having to do repeated calculations of y values?

Answer: Yes, for most functions $f(x)$ it is possible to obtain *formulas* for the sums that give L_n and R_n. **However, the math involved in obtaining those formulas is above the level of MATH 1350.** (Some of the techniques are discussed in the more advanced calculus course MATH 2301, but only for some simple cases.)

Question: Can we figure out the limit $\lim_{n\to\infty} L_n$ analytically, without having to use a computer to find L_n and R_n for larger and larger values of n?

Answer: Yes. If one had the general formula for the value of a particular Riemann sum L_n or R_n, then it would be possible to find the limit $\lim_{n\to\infty} L_n$ or $\lim_{n\to\infty} R_n$ using our limit laws.

Neither of those answers is satisfying for us in MATH 1350. We would like to be able to find the value of definite integrals using analytic techniques at the level of our course.

Is There an Easier Way to Compute Definite Integrals Analytically?

For a general function $f(x)$ whose graph is not made up of basic geometric shapes, we want an analytic way to find

$$Signed\ Area = SA = \int_{x=a}^{x=b} f(x)dx \stackrel{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

but the computation of those limits analytically depends on finding formulas for the Riemann sums that give L_n and R_n, something that is beyond the level of this class.

Question: What do we do? Is there another way, an easier way, to find this value?

$$Signed\ Area = SA = \int_{x=a}^{x=b} f(x)dx \stackrel{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n =?$$

Answer: We will see the answer to that question in the next video.