Subject for this video:

The Fundamental Theorem of Calculus

Reading:

- **General:** Section 5.5 The Fundamental Theorem of Calculus
- **More Specifically:** Pages 369 - 372, Examples 1,2

Homework: H78: Basic Definite Integrals Using the Fundamental Theorem of Calculus (5.5#13,15,17,19,21,23,27,29,31,35,36*)

Recall the Definition of the Definite Integral from Section 5.4 and from a previous video

Definition of the *Definite Integral and Signed Area*

Words: The definite integral of $f(x)$ from a to b.

Symbol: $\int_a^b f(x)dx$

Alternate Words: The *signed area* of the region between the graph of $f(x)$ and the x axis on the interval $[a,b]$.

Alternate Symbol: SA

Usage: $f(x)$ is continuous on the interval $[a,b]$.

Meaning: the number $\lim_{n\to\infty} L_n$ (which is also the value of $\lim_{n\to\infty} R_n$)

That is,

$$SA = \int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

Additional Terminology: The number a is called the *lower limit of the integration*. The number b is called the *upper limit of the integration*.

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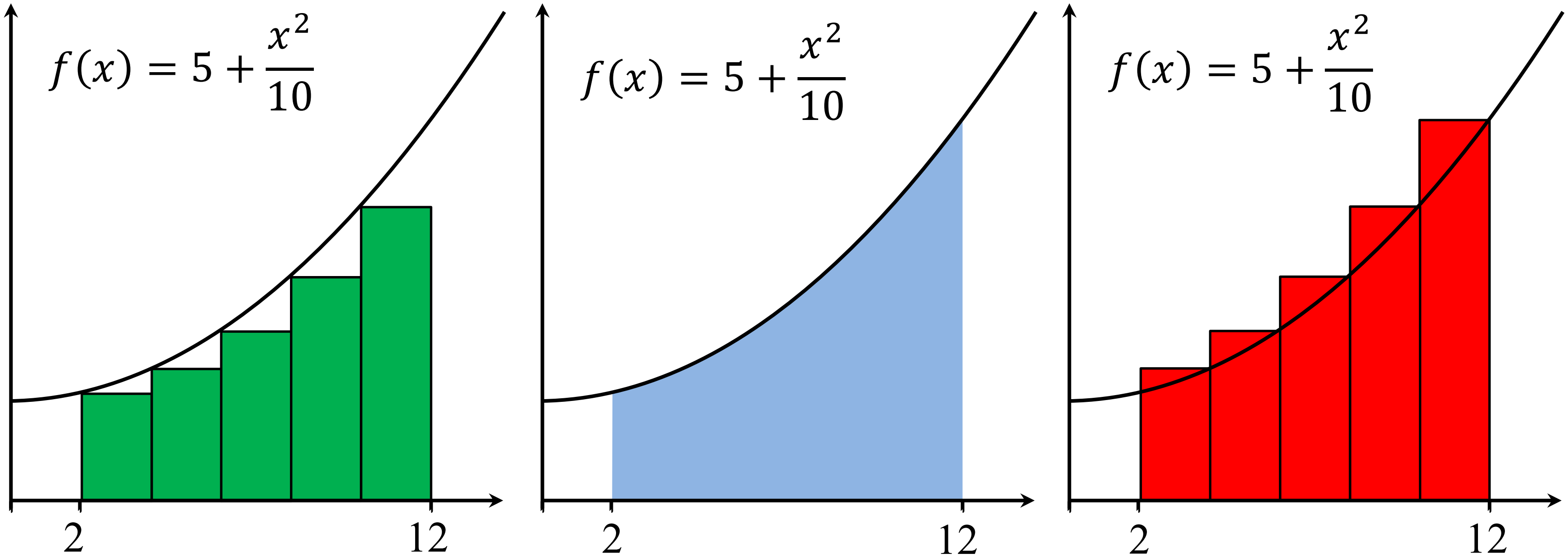
Recall that in a previous video, we used rectangles to estimate the signed area of a region.

The region was between the graph of $f(x) = 5 + \frac{x^2}{10}$ and the x axis from $x = 2$ to $x = 12$



L_n $\qquad SA = \int_2^{12} 5 + \frac{x^2}{10}\,dx \qquad$ R_n

Because the function $f(x) = 5 + \frac{x^2}{10}$ is increasing on the interval [2,12], we know that

$$L_n < SA = \int_2^{12} 5 + \frac{x^2}{10}\,dx < R_n$$

Left and Right Riemann Sums for $f(x) = 5 + \frac{x^2}{10}$ on the interval $[2, 12]$.

n	L_n	SA (unknown)	R_n
5	$L_5 = 94$	SA (unknown)	$R_5 = 122$
10	$L_{10} = 100.5$	SA (unknown)	$R_{10} = 114.5$
100	$L_{100} = 106.635$	SA (unknown)	$R_{100} = 108.035$
1000	$L_{1000} = 107.263$	SA (unknown)	$R_{1000} = 107.403$
10000	$L_{10000} = 107.3263$	SA (unknown)	$R_{10000} = 107.3403$
⋮	⋮	⋮	⋮
$n \to \infty$	L_n getting closer to around 107.33	SA (unknown)	R_n getting closer to around 107.33

Based on the table, we can see that unknown value SA of the area of the region between the graph of $f(x) = 5 + \frac{x^2}{10}$ and the x axis on the interval [2,12], must be around

$$SA = \int_2^{12} 5 + \frac{x^2}{10} dx \approx 107.33$$

.

We have only seen that *one* example of the definition definition of signed area (the definite integral) in use. In that example, we have used a *computer* to find L_n and R_n for particular values of n, as n got larger and larger, and we *guessed* at a *rough value* for the limit. That is, we guessed at a rough value for the value of the definite integral.

We discussed these obvious questions:

Question: Can we do the Riemann Sums L_n and R_n *analytically*, obtaining *general formulas* instead of having to do repeated calculations of y values?

Answer: Yes, for most functions $f(x)$ it is possible to obtain *formulas* for the sums that give L_n and R_n. **However, the math involved in obtaining those formulas is above the level of MATH 1350.**

Question: Can we figure out the limit $\lim_{n\to\infty} L_n$ analytically, without having to use a computer to find L_n and R_n for larger and larger values of n?

Answer: Yes. If one had the general formula for the value of a particular Riemann sum L_n or R_n, then it would be possible to find the limit $\lim_{n\to\infty} L_n$ or $\lim_{n\to\infty} R_n$ using our limit laws.

Neither of those answers is satisfying for us in MATH 1350. We would like to be able to find the value of definite integrals using analytic techniques at the level of our course.

Is There an Easier Way to Compute Definite Integrals Analytically?

For a general function $f(x)$ whose graph is not made up of basic geometric shapes, we want an analytic way to find

$$Signed\ Area = SA = \int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

but the computation of those limits analytically depends on finding formulas for the Riemann sums that give L_n and R_n, something that is beyond the level of this class.

Question: What do we do? Is there another way, an easier way, to find this value?

$$Signed\ Area = SA = \int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n =?$$

That brings us to Section 5.5 The Fundamental Theorem of Calculus

We begin our discussion for Section 5.5 by recalling *notation* that we have used for integrals.

We have seen *two* uses of the integration symbol.

The *Indefinite Integral*

Symbol: $\int f(x)dx$

Spoken: the ***indefinite integral*** of $f(x)$

Meaning: the ***general antiderivative*** of $f(x)$

The *Definite Integral and Signed Area*

Symbol: $\int_a^b f(x)dx$

Spoken: The ***definite integral*** of $f(x)$ from a to b.

Informal meaning, in terms of the graph: The *signed area* of the region between the graph of $f(x)$ and the x axis on the interval $[a, b]$.

The two concepts (*antiderivative* and *signed area*) seem unrelated. It should seem strange that such *similar-looking* symbols would be used to denote such *dissimilar* mathematical concepts.

But it turns out that there is a very important relationship between the two concepts. The relationship is called the *Fundamental Theorem of Calculus*.

The *Fundamental Theorem of Calculus* (*FTC*)

(the relationship between ***definite integrals*** and ***antiderivatives***)

If $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Interpreting the Statement of the Fundamental Theorem of Calculus

We should carefully parse the symbols in the Fundamental Theorem to fully understand what it says:

Given a function $f(x)$ that is continuous on an interval $[a, b]$, there are two processes that one could run to get a number:

Process #1 that leads to a number: Use the *Definite Integral*.

Define the result of Process #1 to be the number

$$SA = \int_a^b f(x)dx$$

In Section 5.4 (discussed in previous videos), we had a definition of what this signed area means, a definition of how to get its value using a limit of a Riemann sum, but we had no practical way to compute its value analytically. Using MATH 1350-level skills, we can only get the signed area analytically in certain simple cases when the graph of $f(x)$ is made up of basic geometric shapes/

Process #2 that leads to a number: Use the *Indefinite Integral*.

The indefinite integral can be used to find an *antiderivative* of $f(x)$. This antiderivative is a *function* that could be called $F(x)$.

$$F(x) + C = \int f(x)dx$$

One can substitute the endpoints of the interval into $F(x)$ to get two numbers $F(a)$ and $F(b)$. Define the result of Process #2 to be the number $F(b) - F(a)$

The *Fundamental Theorem of Calculus* (*FTC*) tells us that the numbers obtained by Process #1 and Process #2 are equal:

$$\underbrace{\int_a^b f(x)dx}_{Process\ \#1} \underset{FTC}{=} \underbrace{F(b) - F(a)}_{Process\ \#2}$$

The usefulness of the Fundamental Theorem is that it answers the question posed at the beginning of this video. That is,

We are often interested in the number that is the result of Process #1 (because it represents a *signed area* that we want to find), but we generally have no way of finding that number analytically (because it would involve difficult techniques beyond the level of this course.)

But Process #2 is straightforward: We can use our Indefi*nite Integral Rules* to find an antiderivative $F(x)$, and then substitute in $x = a$ and $x = b$ to get two numbers $F(a)$ and $F(b)$, and then subtract to get $F(b) - F(a)$.

The Fundamental Theorem tells us that the number that we get from Process #2 equals the number that would be obtained from Process #1.

Our first test of the Fundamental Theorem will be to use it on a function where we actually know how to find the number from Process #1. That is, a function for which we know how to find the signed area using geometry.

[Example 1] Verifying the Fundamental Theorem for a function with a simple geometric graph.

The goal is to find the value of the definite integral $SA = \int_1^5 2x - 4dx$ using two methods.

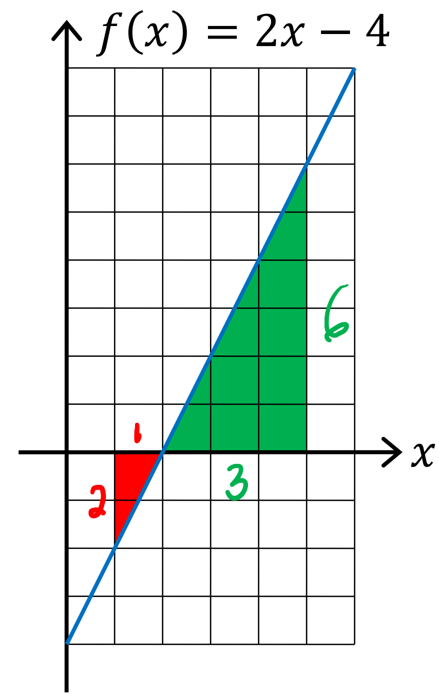
(a) Find the definite integral by using Process #1.

That is, interpret the definite integral as a *signed area* and use *geometry* to find that area.

Solution:

We draw the graph of $f(x) = 2x - 4$ from $x = 0$ to $x = 6$.

Then we shade the region between the graph of $f(x)$ and the x-axis the interval [1,5]. The shaded region is made up of two right triangles.



The red triangle has $b = 1$, $h = 2$, and area $A = \frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot 2 = 1$.

The green triangle has $b = 3$, $h = 6$, and area $A = \frac{1}{2}bh = \frac{1}{2} \cdot 3 \cdot 6 = 9$.

The signed area is $SA = -red + green = -1 + 9 = 8$

So the value of the integral is

$$SA = \int_1^5 2x - 4dx \underset{\substack{using \\ geometry}}{=} 8$$

(b) Find the definite integral using Process #2

That is, first find the general antiderivative by finding the indefinite integral

$$F(x) = \int 2x - 4dx$$

and then find $F(5) - F(1)$.

Solution

integrand $f(x) = 2x - 4 = 2x - 4 \cdot 1$

First we find the indefinite integral

$n=1$, $n=0$, $1 = x^0$

$$F(x) = \int 2x - 4dx = 2\int xdx - 4\int 1dx + C = 2\left(\frac{x^{1+1}}{1+1}\right) - 4\left(\frac{x^{0+1}}{0+1}\right) + C = x^2 - 4x + C$$

Power rule

Notice that the indefinite integral is a *function form*, and so it has a *constant of integration.*

Next, we find $F(5) - F(1)$.

$$\begin{aligned} F(5) - F(1) &= ((5)^2 - 4(5) + C) - ((1)^2 - 4(1) + C) \\ &= (25 - 20 + C) - (1 - 4 + C) \\ &= (5 + C) - (-3 + C) \\ &= 8 \end{aligned}$$

Notice that $F(5) - F(1)$ is a *number*, and it does not contain any unknown constants.

(c) Compare results of (a) and (b)

$$\underbrace{\int_1^5 2x - 4dx}_{Process\ \#1} = 8 = \underbrace{F(5) - F(1)}_{Process\ \#2}$$

Observe that the results of Process #1 and Process #2 are equal, as predicted by the Fundamental Theorem of Calculus.

End of [Example 1]

Our first example was a success: it confirmed what the *Fundamental Theorem of Calculus* says, that the numbers obtained by these two processes are equal.

The number that is the result of Process #1 = The number that is the result of Process #2

$$\underbrace{\int_a^b f(x)dx}_{Process\ \#1} \underset{FTC}{=} \underbrace{F(b) - F(a)}_{Process\ \#2}$$

But remember, the importance of the *Fundamental Theorem of Calculus* is that for many functions, Process #1 is not possible using MATH 1350-level skills, while process #2 is possible.

Later in this video, we will consider more examples where we use the *Fundamental Theorem of Calculus*. But before doing that, it is useful to introduce some new notation.

Definition of *"evaluated at" notation*

Symbol: $F(x)|_a$

Spoken: F *evaluated at* a

Meaning: The number $F(a)$

[Example 2] For $F(x) = 5x + \frac{x^3}{30}$ find $F(x)|_2$

Solution:

$$F(x)|_2 = F(2) = 5(2) + \frac{(2)^3}{30} = 10 + \frac{8}{30} = \frac{308}{30}$$

End of [Example 2]

Definition of *change in F notation*

Symbol: $F(x)|_a^b$

Spoken: *The change in F from* a *to* b

Meaning: The number $F(b) - F(a)$

[Example 3] For $F(x) = 5x + \frac{x^3}{30}$ find $F(x)|_2^{12}$

Solution:

$$\begin{aligned} F(x)|_2^{12} &= F(12) - F(2) \\ &= \left(5(12) + \frac{(12)^3}{30}\right) - \left(5(2) + \frac{(2)^3}{30}\right) \\ &= \left(60 + \frac{1728}{30}\right) - \frac{308}{30} \\ &= \left(\frac{1800}{30} + \frac{1728}{30}\right) - \frac{308}{30} \\ &= \left(\frac{3528}{30}\right) - \frac{308}{30} \\ &= \frac{3220}{30} \\ &= \frac{322}{3} \end{aligned}$$

End of [Example 3]

Rewrite the *Fundamental Theorem of Calculus* using *change in F notation*.

The right side of the Fundamental Theorem is the expression

$$F(b) - F(a)$$

Notice that this expression can be rewritten, using *change in F notation*, as

$$F(b) - F(a) = F(x)|_a^b$$

This can be used to rewrite the *Fundamental Theorem of Calculus* in a new way.

$$\int_a^b f(x)dx \underset{FTC}{=} F(b) - F(a) = F(x)|_a^b = \left(\int f(x)dx\right)\Bigg|_a^b$$

This notation helps us make more sense of the Fundamental Theorem of Calculus, because it reminds us of what we must do when we use the theorem. It also helps remind us of what the Fundamental Theorem is about: it is about the relationship between the *Definite Integral* and the *Indefinite Integral*.

This new way of writing the Fundamental Theorem is so useful that it is worth presenting in a nice green box.

The *Fundamental Theorem of Calculus* (*FTC*)

(the relationship between ***definite integrals*** and ***antiderivatives***)

If $f(x)$ is continuous on the interval $[a, b]$, then

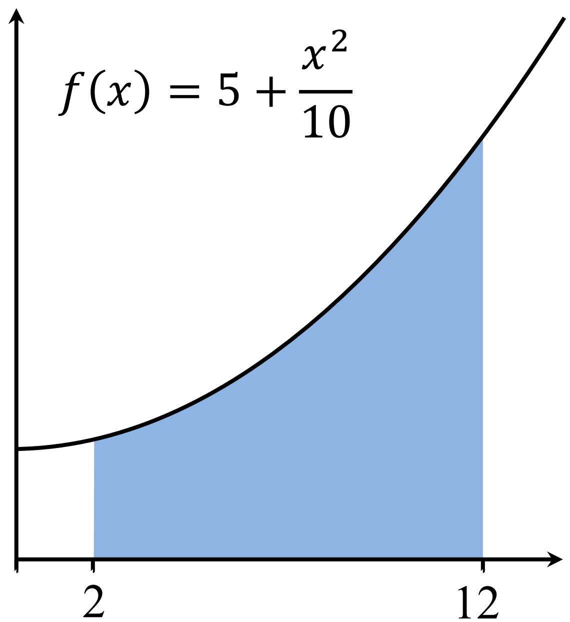
$$\int_a^b f(x)dx \underset{FTC}{=} \left(\int f(x)dx\right)\Big|_a^b$$

For the rest of this video, we will use the Fundamental Theorem of Calculus in examples.

[Example 4] Revisiting earlier example

Use the *Fundamental Theorem of Calculus* to find $\int_2^{12} 5 + \frac{x^2}{10}\, dx$

Recall that the definite integral is defined to be the value of the blue shaded area.



$$SA = \int_2^{12} 5 + \frac{x^2}{10}\, dx$$

Previously, we used a computer to *estimate* the signed area, and we guessed that $SA \approx 107.33$

Solution:

$$\int_2^{12} 5+\frac{x^2}{10}\,dx \underset{FTC}{=} \left(\int 5+\frac{x^2}{10}\,dx\right)\Bigg|_2^{12}$$

$$= \left(5x+\frac{x^3}{30}+C\right)\Bigg|_2^{12}$$

$$= \left(5(12)+\frac{(12)^3}{30}+C\right)-\left(5(2)+\frac{(2)^3}{30}+C\right)$$

$$= \left(5(12)+\frac{(12)^3}{30}\right)-\left(5(2)+\frac{(2)^3}{30}\right)$$

$$= \frac{322}{3} \quad \text{used result of [Example 3]}$$

$$= 107.\overline{333}$$

Indefinite Integral Details

$$F(x) = \int 5+\frac{x^2}{10}\,dx$$

$$= 5\int 1\,dx + \frac{1}{10}\int x^2\,dx$$

use Power Rule with n=0, n=2

$$= 5\left(\frac{x^1}{1}\right)+\frac{1}{10}\left(\frac{x^3}{3}\right)+C$$

$$= 5x+\frac{x^3}{30}+C$$

Remark: This is an *exact* answer. Our previous approximation, $SA \approx 107.33$, was pretty good!

End of [Example 4]

[Example 5] Use the *Fundamental Theorem of Calculus* to find $\int_2^3 \frac{7}{x^2}\,dx$

(Give an answer that is an integer or a simplified fraction.)

$$\int_2^3 \frac{7}{x^2}\,dx \underset{FTC}{=} \left(\int \frac{7}{x^2}\,dx\right)\Big|_2^3$$

$$= \left(-\frac{7}{x} + C\right)\Big|_2^3$$

$$= \left(-\frac{7}{3} + C\right) - \left(-\frac{7}{2} + C\right)$$

$$= -\frac{7}{3} + \frac{7}{2}$$

$$= \frac{-14}{6} + \frac{21}{6}$$

$$= \frac{7}{6}$$ exact answer

≈ 1.667 rounded to 3 decimal places.

Indefinite Integral Details

rewrite the integrand

$$f(x) = \frac{7}{x^2} = 7 \cdot x^{-2}$$

Now integrate

$$F(x) = \int f(x)\,dx$$

$$= \int 7 \cdot x^{-2}\,dx$$

$$= 7 \int x^{-2}\,dx \quad \leftarrow n = -2$$

$$= 7\left(\frac{x^{-2+1}}{-2+1}\right) + C$$

$$= 7\left(\frac{x^{-1}}{-1}\right) + C$$

$$= -\frac{7}{x} + C$$

[Example 6] Use the *Fundamental Theorem of Calculus* to find $\int_2^{10} \frac{3}{x}\,dx$

Solution:

$$\int_2^{10} \frac{3}{x}\,dx \underset{FTC}{=} \left(\int \frac{3}{x}\,dx\right)\Big|_2^{10}$$

$$= (3\ln(|x|) + C)|_2^{10}$$

$$= (3\ln(|10|) + C) - (3\ln(|2|) + C)$$

$$= 3\ln(|10|) - 3\ln(|2|)$$

$$= 3\ln(10) - 3\ln(2)$$

$\ln(a) - \ln(b) = \ln\left(\frac{b}{a}\right)$

$$= 3\ln\left(\frac{10}{2}\right)$$

$$= 3\ln(5) \quad (\textit{exact answer})$$

$$\approx 4.828 \quad (\textit{decimal approximation})$$

Indefinite Integral Details

$$F(x) = \int \frac{3}{x}\,dx$$

$$= 3\int \frac{1}{x}\,dx + C$$

$$= 3\ln(|x|) + C$$

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[Example 7] Use the *Fundamental Theorem of Calculus* to find $\int_0^1 5 \cdot \sqrt[5]{x}\,dx$

(Give an answer that is an integer or a simplified fraction.)

$$\int_0^1 5\cdot\sqrt[5]{x}\,dx \underset{\text{FTC}}{=} \left(\int 5\cdot\sqrt[5]{x}\,dx\right)\Big|_0^1$$

$$= \left(\frac{25}{6}x^{6/5}+C\right)\Big|_0^1$$

$$= \left(\frac{25}{6}(1)^{6/5}+C\right)-\left(\frac{25}{6}(0)^{6/5}+C\right)$$

$$= \left(\frac{25}{6}(1)\right)-\left(\frac{25}{6}(0)\right) \rightarrow 0$$

$$= \frac{25}{6}$$ simplified fraction

Indefinite Integral Details

Rewrite the integrand first

$$f(x) = 5\cdot\sqrt[5]{x} = 5\cdot x^{1/5}$$

Now integrate

$$F(x) = \int f(x)\,dx$$

$$= \int 5\cdot x^{1/5}\,dx$$

$$= 5\int x^{1/5}\,dx$$

$$= 5\left(\frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1}\right)+C$$

$$= 5\left(\frac{x^{6/5}}{\frac{6}{5}}\right)+C$$

$$= 5\cdot\frac{5}{6}x^{6/5}+C$$

$$= \frac{25}{6}x^{6/5}+C$$

[Example 8] Use the *Fundamental Theorem of Calculus* to find $\int_0^2 e^{(3x)}\,dx$

(Give an exact answer and a decimal approximation, rounded to three decimal places.)

$$\int_0^2 e^{(3x)}\,dx \underset{FTC}{=} \left(\int e^{(3x)}\,dx\right)\Big|_0^2$$

Indefinite Integral Details

$$\int e^{(3x)}\,dx = \frac{e^{(3x)}}{3} + C$$

$$= \left(\frac{e^{(3x)}}{3} + C\right)\Big|_0^2$$

$$= \left(\frac{e^{(3(2))}}{3} + C\right) - \left(\frac{e^{(3(0))}}{3} + C\right)$$

$$\frac{d}{dx} e^{(kx)} = k \cdot e^{(kx)}$$

$$\int e^{(kx)}\,dx = \frac{e^{(kx)}}{k} + C$$

$$= \frac{e^6}{3} - \frac{e^0}{3}$$

$$= \frac{e^6}{3} - \frac{1}{3}$$

$$= \frac{e^6 - 1}{3} \approx 134.143$$

exact answer

decimal approximation

[Example 9] Use the *Fundamental Theorem of Calculus* to find $\int_{12}^{2} 5 + \frac{x^2}{10}\, dx$

(Give an answer that is an integer or a simplified fraction.)

$$\int_{12}^{2} 5 + \frac{x^2}{10}\, dx = -\int_{2}^{12} 5 + \frac{x^2}{10}\, dx = -\frac{322}{3} = -107.\overline{333}$$

Property: $\int_{a}^{b} f(x)\, dx = -\int_{b}^{a} f(x)\, dx$

use result of [Example 4]

[Example 10] Use the *Fundamental Theorem of Calculus* to find $\int_5^5 (285 - 17x + 23x^2)^{13}\,dx$

(Give an answer that is an integer or a simplified fraction.)

Integral equals zero, because of property of integrals

$$\int_a^a f(x)\,dx = 0$$