Subject for this video:

Using the Definite Integral to Solve Total Change Problems

Reading:

- **General:** Section 5.5 The Fundamental Theorem of Calculus
- **More Specifically:** Pages 374 - 375, Examples 5, 6

Homework: H80: Using the Definite Integral to Solve Total Change Problems (5.5#69,70,89)

Recall the Fundamental Theorem of Calculus as it was first presented in the video for H78

The *Fundamental Theorem of Calculus* (*FTC*)

(the relationship between ***definite integrals*** and ***antiderivatives***)

If $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Later in that video, *change in F notation* was used rewrite the *Fundamental Theorem of Calculus*.

$$\int_a^b f(x)dx \underset{FTC}{=} F(b) - F(a) = F(x)|_a^b = \left(\int f(x)dx\right)\Big|_a^b$$

Total Change Problems

We start by discussing the fact that the *Fundamental Theorem of Calculus* can be written in different forms.

Consider the first presentation of the Fundamental Theorem:

$$\int_a^b f(x)dx \underset{FTC}{=} F(b) - F(a)$$

In this expression, the integrand is a function $f(x)$, and the function on the other side is $F(x)$ which is an antiderivative of $f(x)$. In other words, $F'(x) = f(x)$.

Because $F'(x) = f(x)$, we can rewrite the Fundamental Theorem using $F'(x)$ as the integrand, instead of $f(x)$.

$$\int_a^b F'(x) \underset{FTC}{=} F(b) - F(a)$$

It turns out that this form is useful for what are called *Total Change Problems*. Notice that the right side of the equation represents a *change in the value* of $F(x)$. We will denote that change by the capital Greek *Delta* symbol, ΔF.

$$F(b) - F(a) = \Delta F = change\ in\ F$$

We see that the Fundamental Theorem of Calculus tells us that

the change in the value of a function = the definite integral of the derivative of the function

There are situations where one knows the *derivative* of a function and wants to know the *change in the value* of the function. In *application* problems (that is, problems about applying math to solve some real world problem), the known derivative will be a rate of change of some quantity. One will be seeking the change in value of the quantity. That is what we will call a *Total Change Problem*. We see that the Fundamental Theorem of Calculus gives us a way to solve that kind of problem.

Definition of *Total Change Problems*

The Problem:

- **Given:** the rate of change of some quantity, $F'(x)$, and two numbers a, b with $a \leq b$,
- **Find:** the change in the quantity, $\Delta F = F(b) - F(a)$

Solution to the Problem: Use the Fundamental Theorem of Calculus

$$\Delta F = F(b) - F(a) \underset{FTC}{=} \int_a^b F'(x)$$

In this video, we will study two examples of *Total Change Problems*.

[Example 1](similar to 5.5#69,70) Marginal Cost and Change in Cost

A company makes bikes. The Marginal Cost is $C'(x) = 100 - \frac{x}{5}$ for $0 \leq x \leq 400$ where the variable x repesents the number of bikes made per month (the quantity).

(a) Find the increase in cost going from production level of 200 bikes per month to 300 bikes per month.

Solution

Observe that this problem has the form of a ***Total Change Problem***.

- **We are Given:**
  - the Marginal Cost, $C'(x) = 100 - \frac{x}{5}$ for $0 \leq x \leq 400$
  - and two numbers $x = 200$ and $x = 300$ with $200 \leq 300$~~5~~,
- **We are Asked to Find:** the change in cost, $\Delta C = C(300) - C(200)$

Therefore, we should solve the problem by using the Fundamental Theorem of Calculus

$$\Delta C = C(300) - C(200)$$

$$\underset{FTC}{=} \int_{200}^{300} C'(x)dx$$

$$= \int_{200}^{300} 100 - \frac{x}{5}dx$$

$$\underset{FTC}{=} \left(\int 100 - \frac{x}{5}dx\right)\Bigg|_{200}^{300}$$

$$= \left(100x - \frac{x^2}{10} + K\right)\Bigg|_{200}^{300} \quad (\textit{see details below})$$

$$= \left(100(300) - \frac{(300)^2}{10} + K\right) - \left(100(200) - \frac{(200)^2}{10} + K\right)$$

$$= (21{,}000 + K) - (16{,}000 + K)$$

$$= 5{,}000$$

Indefinite Integral Details

$$C(x) = \int 100 - \frac{x}{5}dx = \int 100dx - \frac{1}{5}\int xdx = 100\left(\frac{x^{0+1}}{0+1}\right) - \frac{1}{5}\left(\frac{x^{1+1}}{1+1}\right) + K$$

$$= 100x - \frac{x^2}{10} + K$$

Conclusion

The change in cost is \$5000.

Observations

(1) Observe that in solving Total Change Problems, the Fundamental Theorem of Calculus (*FTC*) is used *twice*:

- *FTC* is used to equate the ***change in value of the function*** with a ***definite integral of the function***.
- *FTC* is used again to compute the ***definite integral*** using the ***indefinite integral***.

(2) Note that the ***change in Cost*** is equal to the ***definite integral of the Marginal Cost***. This is an example of what was observed earlier about what the Fundamental Theorem of Calculus tells us:

the change in the value of a function = the definite integral of the derivative of the function

(3) Notice that when we find the indefinite integral of the *Marginal Cost*, $C'(x)$, we denote it $C(x)$ and it has a constant of integration $+K$. So $C(x) = 100x - \frac{x^2}{10} + K$ is the *function form* for the *Cost function*. It is not the actual *Cost function*, because we don't know the value of K. But also realize the meaning of K in this setting: the constant of integration $+K$ is the *fixed Cost*, which we have not been given. But we can still determine the *change in cost*, even not knowing the *fixed cost*.

$$C(0) = 100(0) - \frac{(0)^2}{10} + K = K$$

(b) Illustrate the result of the problem using graphs of $C(x)$ and $C'(x)$.

Solution:

The symbols $\Delta C = C(300) - (200)$ represent a *change in height* on the graph of the Cost function.

$$C(x) = 100x - \frac{x^2}{10} + K \text{ for } 0 \le x \le 400$$

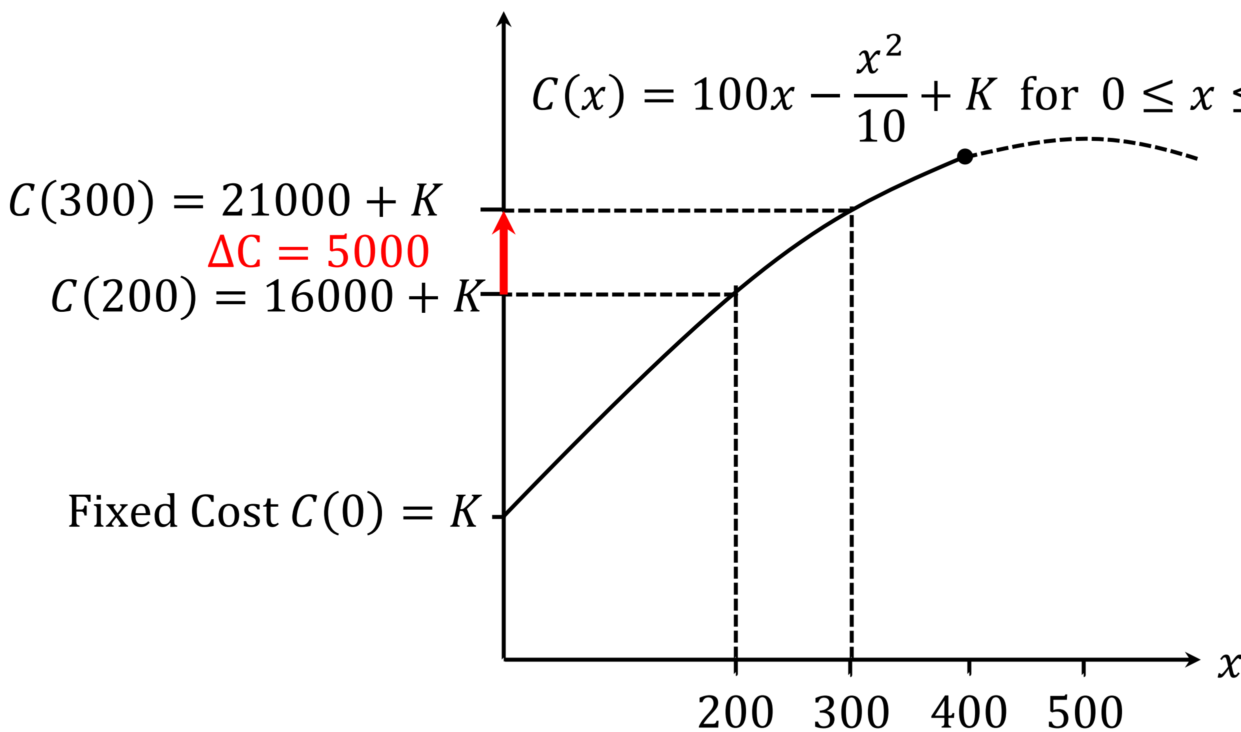
It is worthwhile to think about the shape of the graph of the Cost function. Considered as an abstract mathematical function, $C(x) = 100x - \frac{x^2}{10} + K$ will have a graph that is a *parabola* facing *down*. This means that there will be some value $x = c$ that *maximizes* the value of $C(x)$. Larger or smaller values of x would have smaller corresponding values of $C(x)$. But considered as a cost function, this should seem puzzling, because it would mean that if the company made more than $x = c$ bikes per week, their costs would start going down. This doesn't make sense: making a larger batch of bikes will always be more costly. So how can the Cost function be a parabola facing down?

The key is the domain. Note that we were given a restricted domain $0 \le x \le 400$ for the Marginal Cost $C'(x)$. That means that when we integrate to get the Cost function $C(x)$, it also has a restricted domain $0 \le x \le 400$. It turns out that the max in the function $C(x) = 100x - \frac{x^2}{10} + K$ would occur at $x = 500$. This x value is out of our given domain $0 \leq x \leq 400$. On that restricted domain, the graph of $C(x)$ is only increasing.

The change in cost

$$\Delta C = C(300) - (200)$$

will show up as a change in height on the graph of the Cost function $C(x)$. I have shown it as a red arrow on the graph below. (Note the restricted domain.)



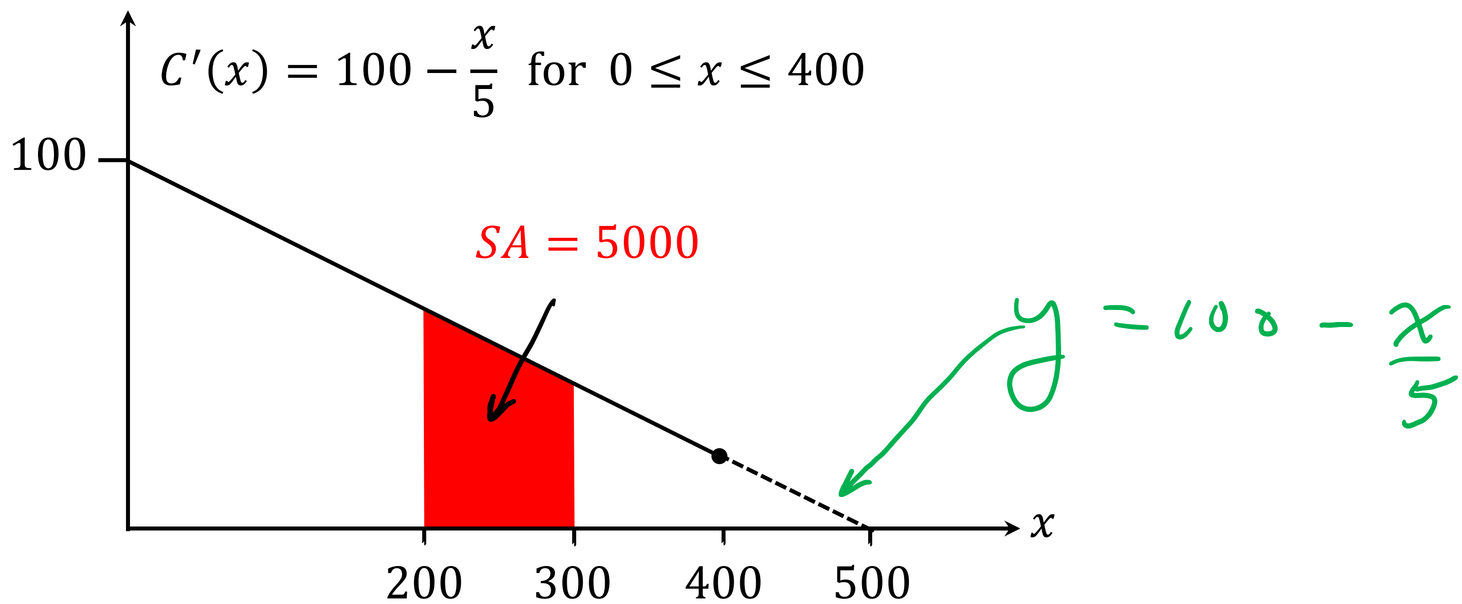
The symbol

$$\int_{200}^{300} C'(t)dt$$

represents the *signed area* of a region on the on the graph of the Marginal Cost function

$$C'(x) = 100 - \frac{x}{5} \ \text{ for } \ 0 \leq x \leq 400$$

That region is shown in red on the graph of $C'(x)$ below. (Note the restricted domain.)



End of [Example 1]

[Example 2] (similar to 5.5#89) Bacteria Growth Rate and Change in Weight

A bacteria culture is growing at a rate

$$W'(t) = .6e^{.2t} \; grams\ per\ hour$$

(a) How much does the weight of the culture change from $t = 5$ hours to $t = 15$ hours?

(Give an exact answer and a decimal approximation)

Solution

Observe that this problem has the form of a ***Total Change Problem***.

- **We are Given:**
  - the rate of change of the weight of the bacteria culture, $W'(t) = .6e^{.2t}$ $grams\ per\ hour$
  - and two numbers $t = 5$ and $t = 15$ with $5 \leq 15$,
- **We are Asked to Find:** the change in the weight of the culture, $\Delta\text{W} = W(15) - F(5)$

Therefore, we should solve the problem by using the Fundamental Theorem of Calculus

$$\Delta W = W(15) - W(5)$$

$$\underset{FTC}{=} \int_{5}^{15} W'(t)\,dt$$

$$= \int_{5}^{15} 0.6e^{(0.2t)}\,dt$$

$$\underset{FTC}{=} \left(\int 0.6e^{(0.2t)}\,dt\right)\Bigg|_{5}^{15}$$

$$= \left(3e^{(0.2t)} + C\right)\Big|_{1}^{5}$$

$$= \left(3e^{(0.2(15))} + C\right) - \left(3e^{(0.2(15))} + C\right)$$

$$= 3e^{(3)} - 3e^{(1)}$$

$$= 3e^{3} - 3e$$

$$\approx 52.1$$

$$\frac{d}{dx} e^{(kx)} = ke^{(kx)}$$

$$\int e^{(kx)}\,dx = \frac{e^{kx}}{k} + C$$

Indefinite Integral Details

$$W(t) = \int W'(t)\,dt$$

$$= \int 0.6e^{(0.2t)}\,dt$$

$$= 0.6 \int e^{(0.2t)}\,dt + C$$

$$= 0.6\left(\frac{e^{(0.2t)}}{0.2}\right) + C$$

$$= 3e^{(0.2t)} + C$$

Conclusion

The change in weight of the culture from $t = 5$ hours to $t = 15$ hours is roughly 52.1 grams.

Observations

(1) Note that the ***change in Weight*** is equal to the ***definite integral of the Growth Rate***. This is an example of what was observed earlier about what the Fundamental Theorem of Calculus tells us:

the change in the value of a function = the definite integral of the derivative of the function

(2) Also, observe that when we find the indefinite integral of the *Growth Rate*, $W'(t)$, we denote it $W(t)$, and observe that it has a constant of integration $+C$. So $W(t) = 3e^{(0.2t)} + C$ is the *function form* for the *Weight function*. It is not the actual *Weight function*, because we don't know the value of C. But also realize the meaning of C in this setting: the constant of integration $+C$ is the *initial weight* of the culture (at time $t = 0$), a weight that we have not been given. But we can still determine the *change in weight*, even not knowing the *initial weight*

The initial weight is

$$W(0) = 3e^{(0.2(0))} + C = 3e^{(0)} + C = 3 \cdot 1 + C$$

$$= 3 + C$$

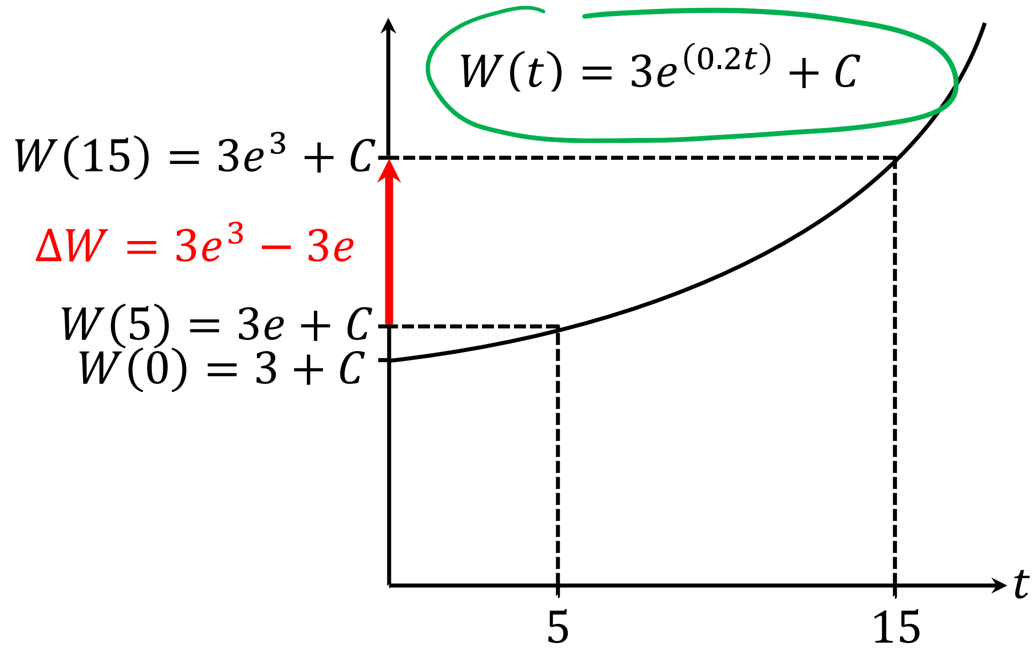
(b) Illustrate the result of the problem using the given graphs of $W(t)$ and $W'(t)$.

Solution:

The symbols

$$\Delta W = W(15) - W(5)$$

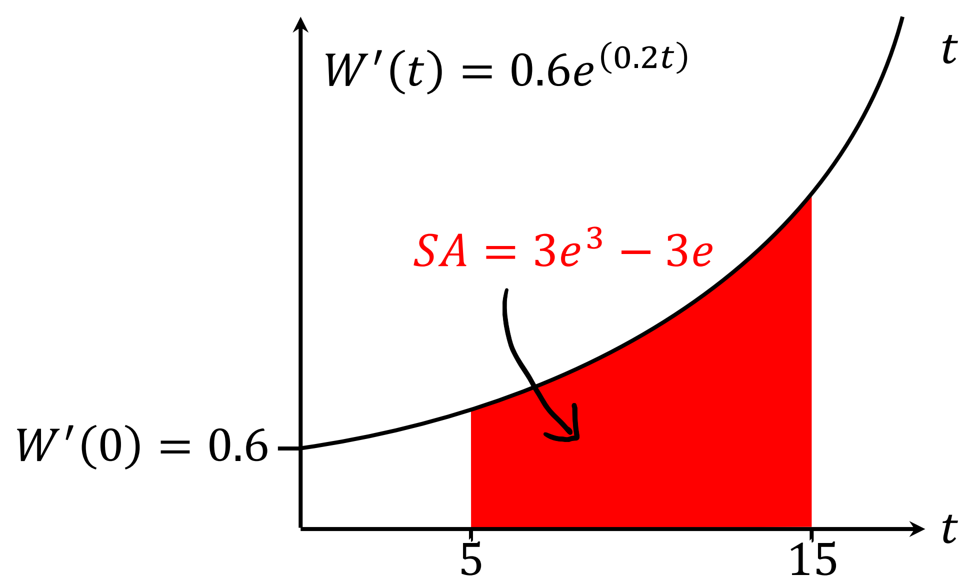
represent a *change in height* on the graph of $W(t)$. That change in height is shown as a red arrow on the graph at right.



The symbol

$$\int_5^{15} W'(t)dt$$

represents the *signed area* of a region on the on the graph of $W'(t)$. That region is shown in red on the graph at right.



End of [Example 2]