Subject for this video:

The Average Value of a Function Over an Interval

Reading:

- **General:** Section 5.5 The Fundamental Theorem of Calculus
- **More Specifically:** Pages 375 - 377, Examples 8,9

Homework: H81: The Average Value of a Function Over an Interval (5.5#49,51,53,93)

Recall the relationship between the Definite Integral and Signed Area.

The *Definite Integral and Signed Area*

Symbol: $\int_a^b f(x)dx$

Spoken: The ***definite integral*** of $f(x)$ from a to b.

Informal meaning, in terms of the graph: The *signed area* of the region between the graph of $f(x)$ and the x axis on the interval $[a, b]$.

And recall the relationship between Definite Integrals and Antiderivatives

The *Fundamental Theorem of Calculus* (*FTC*)

(the relationship between ***definite integrals*** and ***antiderivatives***)

If $f(x)$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)dx \underset{FTC}{=} \left(\int f(x)dx\right)\Bigg|_a^b$$

And recall the Substitution Method shown on the next page.

The Substitution Method for finding the *indefinite integral*

$$F(x) = \int f(x)\,dx$$

where the integrand $f(x)$ involves a *nested function*.

Step 1 Identify the inner function and call it $\boldsymbol{u}$. Write the equation $inner(x) = u$ to introduce the single letter u to represent the inner function. Circle the equation.

Step 2 Build the equation $\boldsymbol{dx = \frac{1}{u'}du}$. To do this, first find u', then use it to build equation $dx = \frac{1}{u'}du$. Circle the equation.

Step 3 Substitute, Cancel, Simplify. In steps (1) and (2) you have two circled equations. Substitute these into the integrand of your indefinite integral. Cancel as much as possible and simplify by using the *Constant Multiple Rule*. The result should be a new basic integral involving just the variable u. (See **Remarks about Step 3** on the next page.)

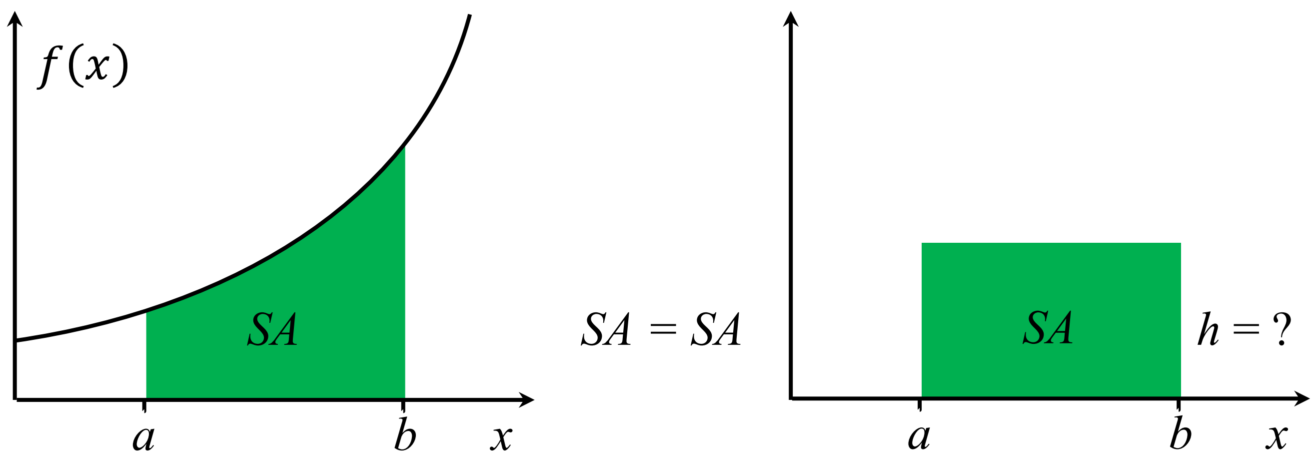
Step 4 Integrate. Find the new indefinite integral by using the indefinite integral rules. The result should be a function involving just the variable u (with constant of integration $+C$).

Step 5 Substitute Back. Substitute $u = inner(x)$ into your function from Step (4) The result will be a new function of just the variable x. (Be sure to include the constant of integration $+C$ in your result.) This is the $F(x)$ that we seek.

The Average Value of a Function Over an Integral

We start with a **geometric question**.

Suppose a given function $f(x)$ is continuous on a given closed interval $[a, b]$. There is a number SA that is the signed area between the graph of $f(x)$ and the x axis on the interval $[a, b]$.
Now suppose that we want to put a rectangle on the same interval $[a, b]$ and we want that rectangle to have the same signed area SA.



Here is our question: What would the height *h* of this rectangle need to be?

The answer is straightforward.

For the region, between the graph of $f(x)$ and the x axis, the signed area is $SA = \int_a^b f(x)dx$

For the rectangle, the signed area is $SA = width \cdot height = (b-a) \cdot h$.

Equating these, we obtain $\int_a^b f(x)dx = (b-a) \cdot h$

Dividing by $(b-a)$ we arrive at the **formula for the height *h***: $h = \frac{1}{(b-a)} \int_a^b f(x)dx$

We have answered our **geometric question**. The answer, the height h, is given a name.

Definition of the *Average Value of a Function Over an Interval*

words: The *Average Value of* $f(x)$ *over the interval* $[a,b]$.

Usage: The function $f(x)$ is continuous on the interval $[a,b]$.

Meaning: The number h given by this formula: $h = \frac{1}{(b-a)} \int_a^b f(x)dx$

Graphical Interpretation: The number h is the height of a rectangle sitting on the interval $[a,b]$ that would enclose a signed area that is equal to the signed area of the graph of f on the same interval.

We will consider three examples about computing the average value of a function on an interval.

[Example 1] (a) Find the average value of $g(x) = 4x + 3x^2$ on the interval $[-3,2]$.

Solution:

We need to find the quantity

$$h = \frac{1}{\left(2 - (-3)\right)} \int_{-3}^{2} 4x + 3x^2 dx$$

The calculation follows on the next page.

Calculation of *h*:

$$h = \frac{1}{(2-(-3))}\int_{-3}^{2} 4x + 3x^2 dx$$

$$\underset{FTC}{=} \frac{1}{5}\left(\int 4x + 3x^2 dx\right)\Big|_{-3}^{2}$$

$$= \frac{1}{5}(2x^2 + x^3 + C)|_{-3}^{2}$$

$$= \frac{1}{5}\left(\left(2(2)^2 + (2)^3 + C\right) - \left(2(-3)^2 + (-3)^3 + C\right)\right)$$

$$= \frac{1}{5}\left((16) - (-9)\right)$$

$$= \frac{1}{5}(25)$$

$$= 5$$

.

Indefinite Integral Details:

$$G(x) = \int 4x + 3x^2 dx$$

$$= 4\int x dx + 3\int x^2 dx$$

n = 1 n = 2

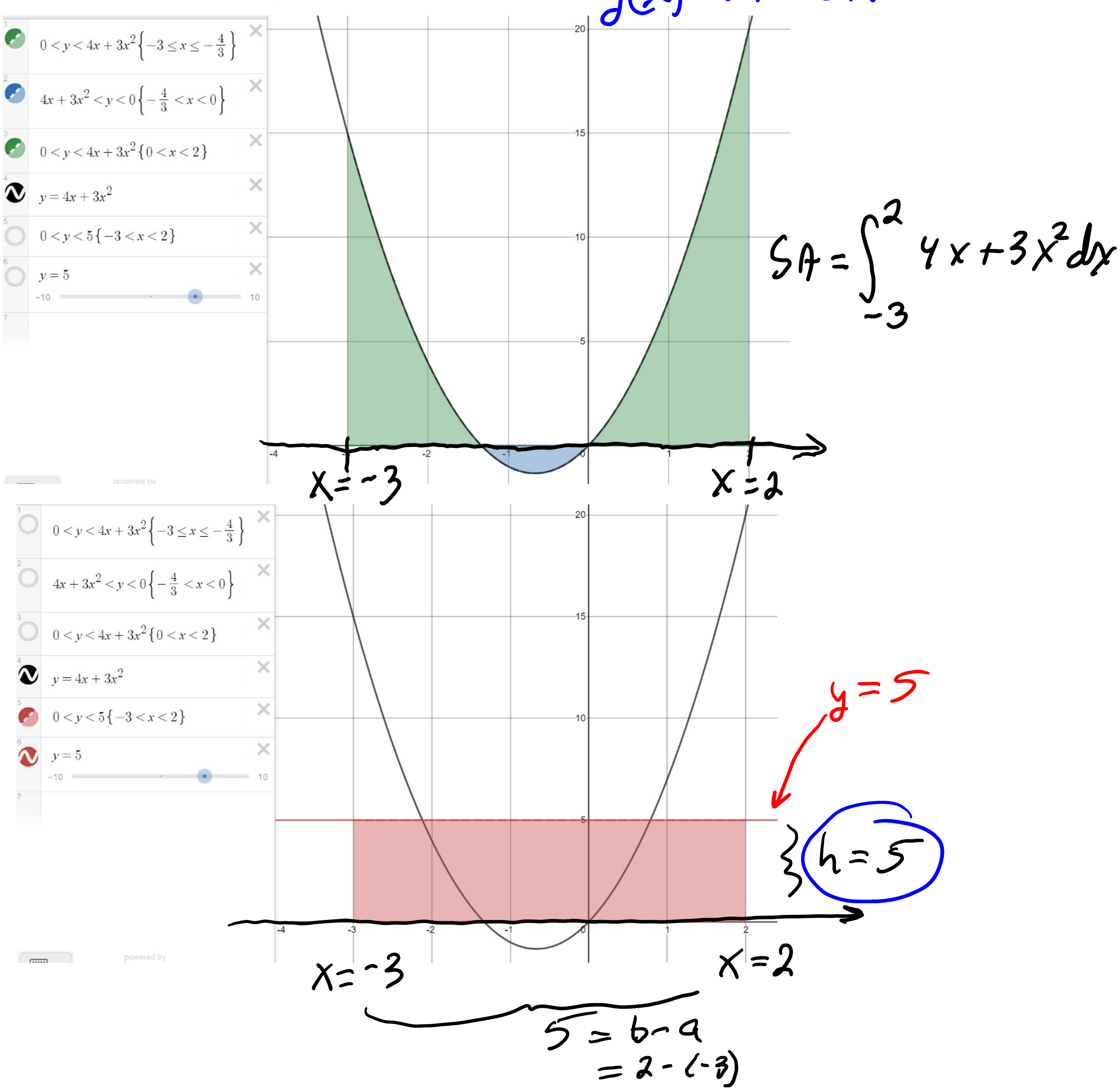
$$= 4\left(\frac{x^{1+1}}{1+1}\right) + 3\left(\frac{x^{2+1}}{2+1}\right) + C$$

$$= 4\left(\frac{x^2}{2}\right) + 3\left(\frac{x^3}{3}\right) + C$$

$$= 2x^2 + x^3 + C$$

.

(b) Illustrate the result using a graph of $g(x)$.



End of [Example 1]

[Example 2] (a) Find the average value of $f(x) = 5\sqrt{x}$ on the interval $[1,16]$.

Solution:

We need to find the quantity

$$h = \frac{1}{(16-1)}\int_{1}^{16} 5\sqrt{x}dx$$

The calculation follows on the next page.

Calculation of *h*

$$h = \frac{1}{(16-1)}\int_1^{16} 5\sqrt{x}dx$$

$$= \frac{1}{3}\int_1^{16} x^{\frac{1}{2}}dx$$

$$\underset{FTC}{=} \frac{1}{3}\left(\int x^{\frac{1}{2}}dx\right)\Bigg|_1^{16}$$

$$= \frac{1}{3}\left(\frac{2x^{\frac{3}{2}}}{3} + C\right)\Bigg|_1^{16}$$

$$= \frac{1}{3}\left(\left(\frac{2(16)^{\frac{3}{2}}}{3} + C\right) - \left(\left(\frac{2(1)^{\frac{3}{2}}}{3} + C\right)\right)\right)$$

$$= \frac{1}{3}\left(\left(\frac{2\cdot 64}{3}\right) - \left(\frac{2\cdot 1}{3}\right)\right)$$

$$= \cdots = 14$$

Indefinite Integral Details:

$$F(x) = \int x^{\frac{1}{2}}dx \quad n = \frac{1}{2}$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

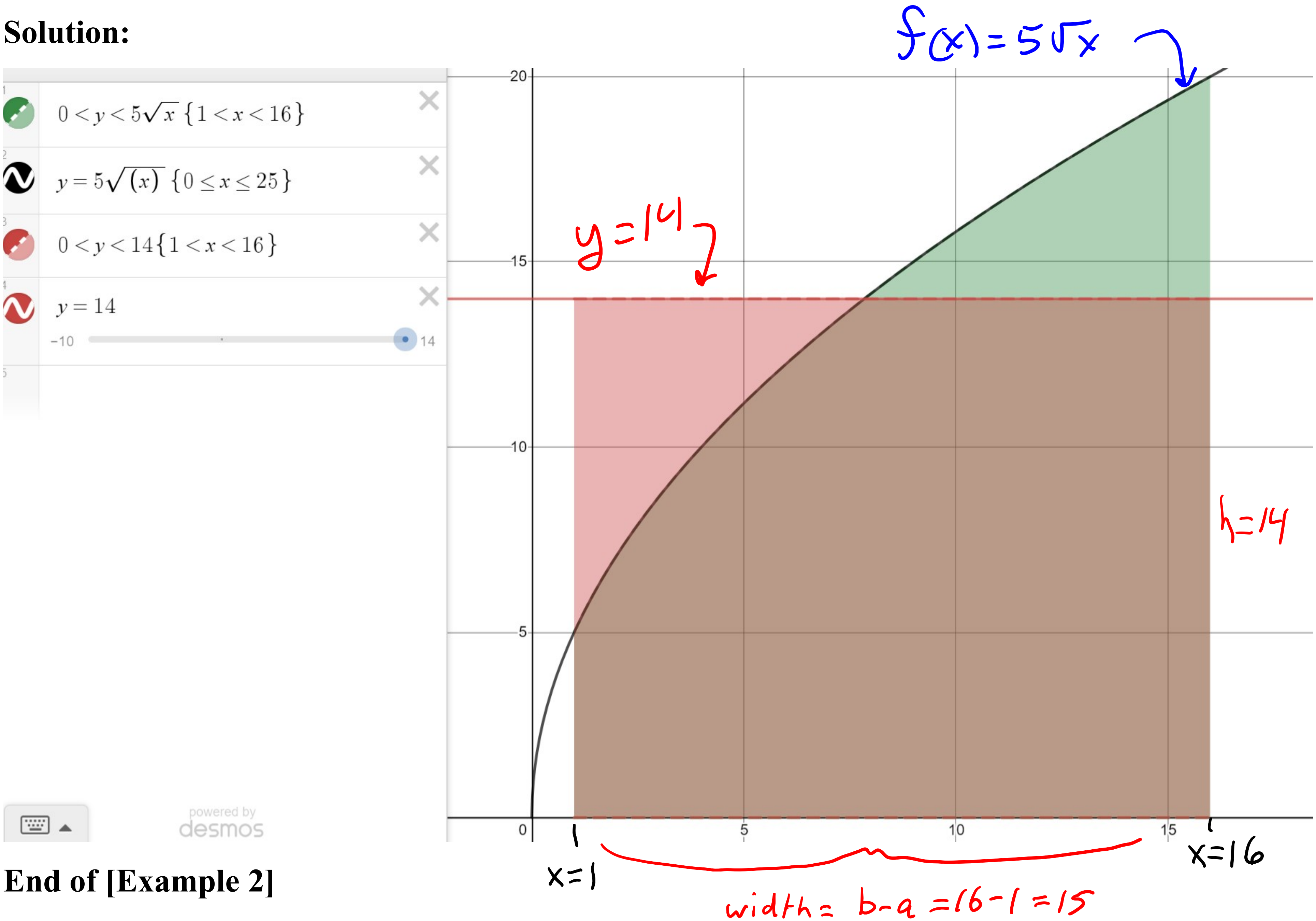
$$F(x) = \frac{2x^{\frac{3}{2}}}{3} + C$$

$$a^{b\cdot c} = (a^b)^c$$

Rational Power Details $(16)^{\frac{3}{2}} = (16)^{\frac{1}{2}\cdot 3} = \left((16)^{\frac{1}{2}}\right)^3 = (4)^3 = 64.$

(b) Illustrate the result using a graph of $f(x)$.

Solution:



End of [Example 2]

[Example 3] (Similar to 5.5#93) A drug is administered to a patient by a pill. The drug concentration in the bloodstream is described by the function

$$C(t) = \frac{0.6t}{t^2 + 9}, \text{ for } 0 \le t \le 24$$

where t is the time in hours after the pill is taken and $C(t)$ is the drug concentration in the bloodstream (in milligrams/liter) at time t.

(a) Find the average drug concentration in the bloodstream over the first 6 hours. Give an exact answer and a decimal approximation.

Solution:

We need to find the quantity

$$h = \frac{1}{(6-0)} \int_0^6 \frac{0.6t}{t^2+9}\,dt = \frac{0.6}{6} \int_0^6 \frac{t}{t^2+9}\,dt = \frac{1}{10} \int_0^6 \frac{t}{t^2+9}\,dt$$

The calculation continues on the next page.

Calculation of *h*:

$$h = \frac{1}{10}\int_0^6 \frac{t}{t^2+9}\,dt$$

$$\underset{FTC}{=} \frac{1}{10}\left(\int \frac{t}{t^2+9}\,dt\right)\Bigg|_0^6$$

see next page

$$= \frac{1}{10}\left(\frac{1}{2}\ln(|t^2+9|) + \cancel{C}\,K\right)\Bigg|_0^6$$

$$= \frac{1}{10}\left(\left(\frac{1}{2}\ln(|(6)^2+9|) + \overset{K}{C}\right) - \left(\frac{1}{2}\ln(|(0)^2+9|) + \overset{K}{C}\right)\right)$$

$$= \frac{1}{10}\cdot\frac{1}{2}\left(\ln(|45|) - (\ln(|9|))\right)$$

$$= \frac{1}{20}\left(\ln(45) - (\ln(9))\right)$$

$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$

$$= \frac{1}{20}\ln\left(\frac{45}{9}\right)$$

$$= \frac{1}{20}\ln(5)$$ exact answer

$$\approx 0.0805$$ decimal approximation

Indefinite Integral Details (Using Substitution Method):

$$Original\ Integral: F(t) = \int \frac{t}{t^2 + 9}\, dt$$

$$Step\ 1: t^2 + 9 = u$$

$$Step\ 2: dt = \frac{1}{2t} du$$

substitute — cancel — simplify

$$Step\ 3: \int \frac{t}{t^2 + 9}\, dt = \int \frac{t}{u} \frac{1}{2t} du = \int \frac{1}{2u} du = \frac{1}{2} \int \frac{1}{u} du$$

$$Step\ 4: \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln(|u|) + C$$

$$Step\ 5: F(t) = \frac{1}{2} \ln(|t^2 + 9|) + C$$

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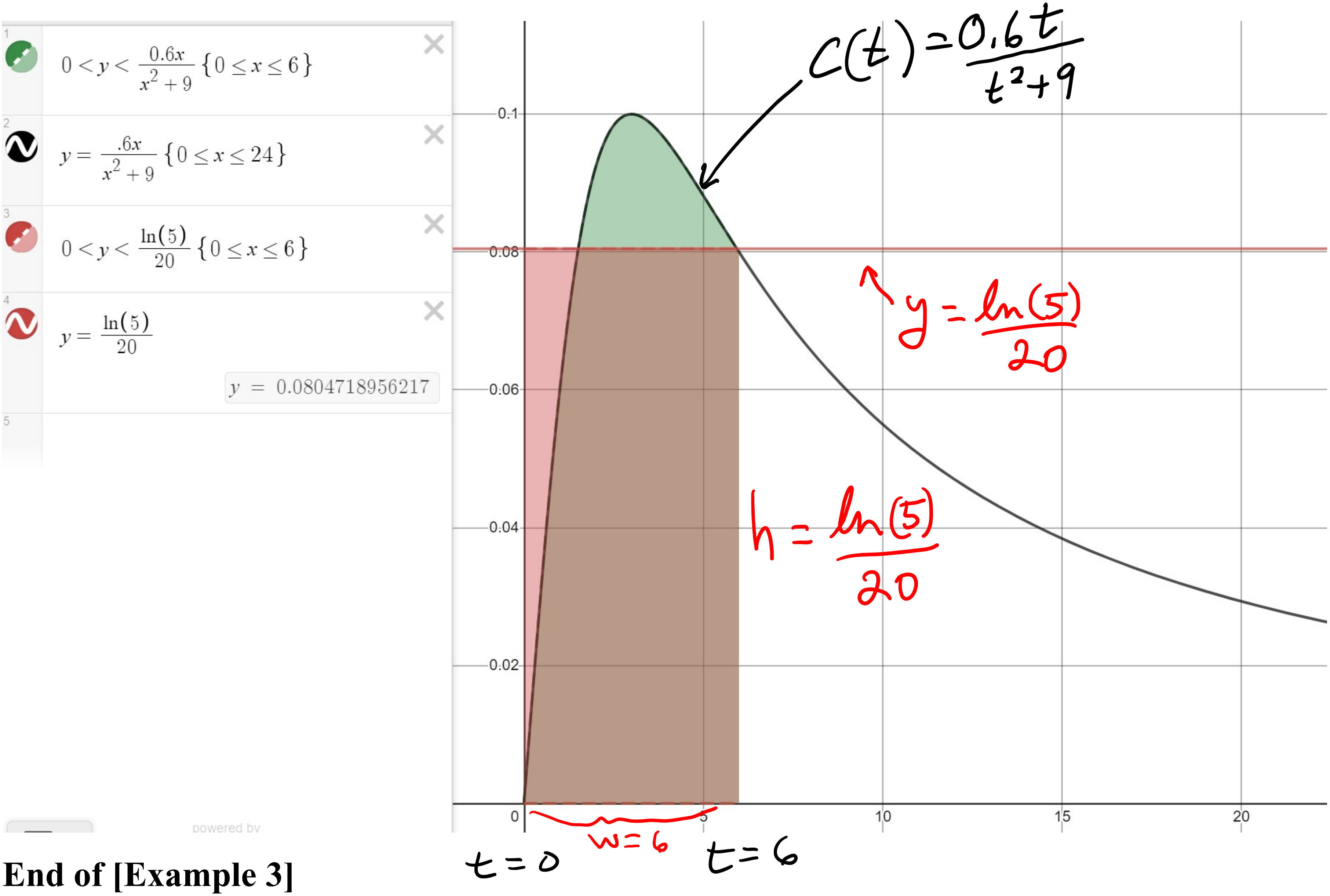
(b) Illustrate the result using a graph of $C(t)$.

Solution:



End of [Example 3]

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