Subject for this video:

The Area Between a Curve and the x Axis

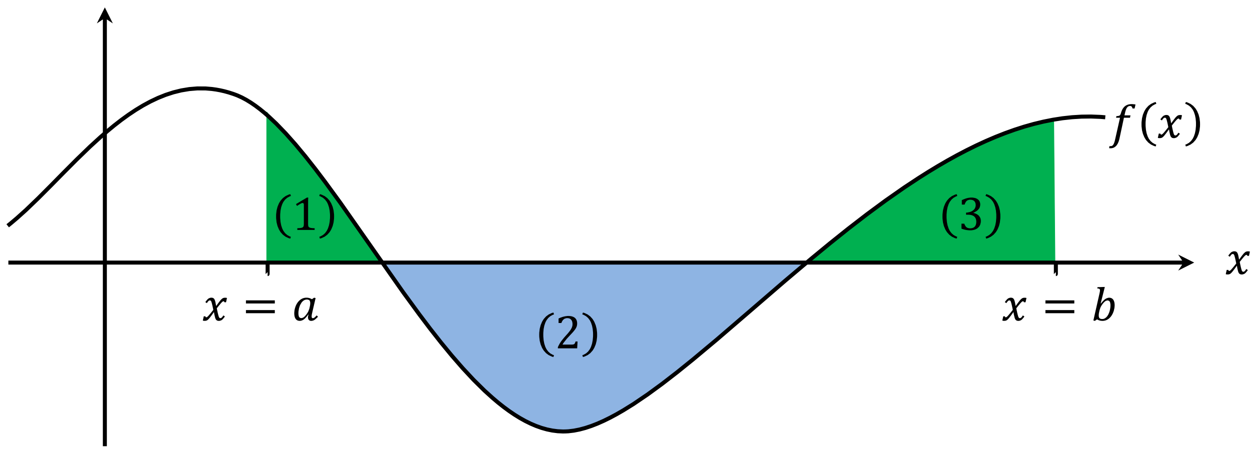
Reading:

- **General:** Section 6.1 The Area Between Curves
- **More Specifically:** Pages 388 – 389 Examples 1,2

Homework: H82: Area between a curve and the x axis (6.1#9,11,17,21,23,25,57)

Recall the Concept of *Unsigned* and *Signed Area*, first introduced in the video for H76.

We discussed two kinds of area created by the graph of a function $\boldsymbol{f(x)}$ and the $\boldsymbol{x}$ axis from $\boldsymbol{x = a}$ to $\boldsymbol{x = b}$.



The two kinds of area were

- ***Unsigned Area*** $(USA) = (1) + (2) + (3)$
- ***Signed Area*** $(SA) = (1) - (2) + (3)$ (Regions under the x axis get a negative sign.)

In Chapter 5 (Sections 5.4 and 5.5), we were primarily interested in *signed area*. We saw the development of the definite integral (defined as the limit of Riemann sums).

The *Definite Integral* refers to *signed area*: $SA = \int_a^b f(x)dx$

But now, in Chapter 6, we are interested in finding what is referred to as ***area between curves***, which is an *unsigned area*. How are we to find unsigned area if our only tool for finding area is the Definite Integral, which finds *signed area*?

The key is to first define what I will call *simple regions*.

Definition of Simple Regions

If $top(x)$ and $bottom(x)$ are continuous functions and $bottom(x) \leq top(x)$ on the interval $[a, b]$, then the region bounded by the four curves

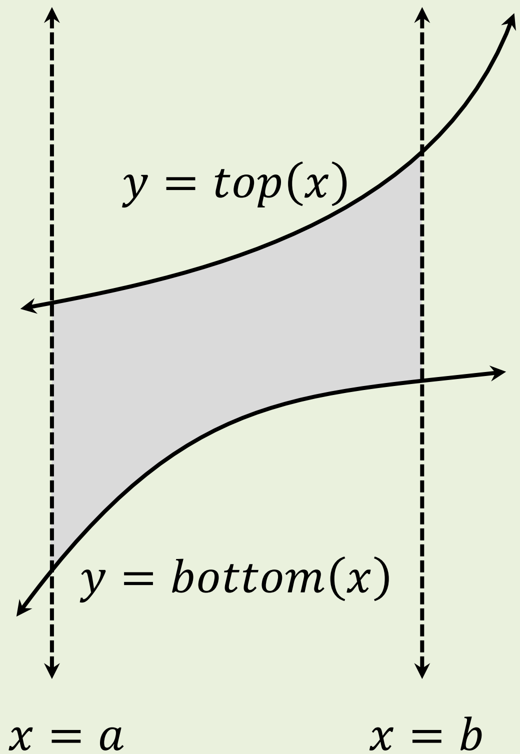
$$y = top(x)$$

$$y = bottom(x)$$

$$x = a$$

$$x = b$$

is what we will call a ***simple region***.



The following theorem explains how one can use the Definite Integral to find the unsigned area between curves for *simple regions*.

Theorem about the Area between Curves for a Simple Region.

If $top(x)$ and $bottom(x)$ are continuous and $bottom(x) \leq top(x)$ on the interval $[a, b]$, then the region bounded by the four curves

$$y = top(x)$$

$$y = bottom(x)$$

$$x = a$$

$$x = b$$

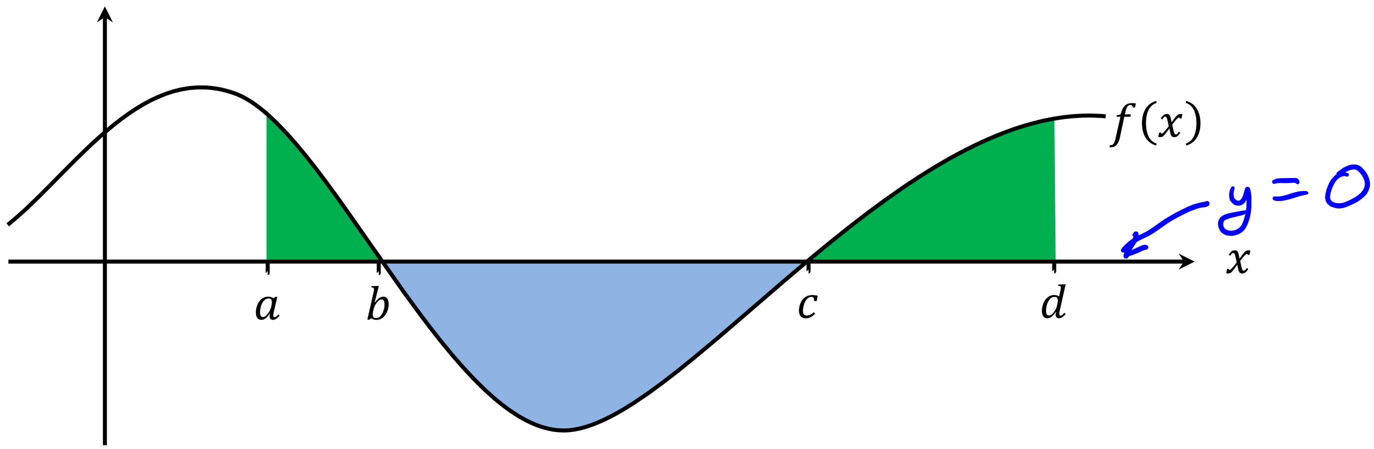
is what we have defined to be called a ***simple region***.

The ***Area Between Curves*** (*unsigned* area) for this region is given by the definite integral

$$USA = \int_a^b top(x) - bottom(x)dx$$

For the rest of this video, we will study examples involving computing the Area Between Curves.

[Example 1](Similar to 6.1#9,11) Function $f(x)$ has graph shown below.



(a) Set up a definite integral that represents the area between the graph of $f(x)$ and the x axis from a to b.

unsigned area

$\text{top}(x) = f(x)$

$\text{bottom}(x) = 0$

$$USA = \int_a^b \text{top}(x) - \text{bottom}(x)\,dx = \int_a^b f(x) - 0\,dx = \int_a^b f(x)\,dx$$

(b) Set up a definite integral that represents the area between the graph of $f(x)$ and the x axis from b to c.

top $(x) = 0$, bottom $(x) = f(x)$

$$USA = \int_b^c \text{top}(x) - \text{bottom}(x)\,dx = \int_b^c 0 - f(x)\,dx = \int_b^c -f(x)\,dx$$

$$= -\underbrace{\int_b^c f(x)\,dx}_{\text{this definite integral gives the signed area, which is negative}}$$

this will be the unsigned area, which is positive

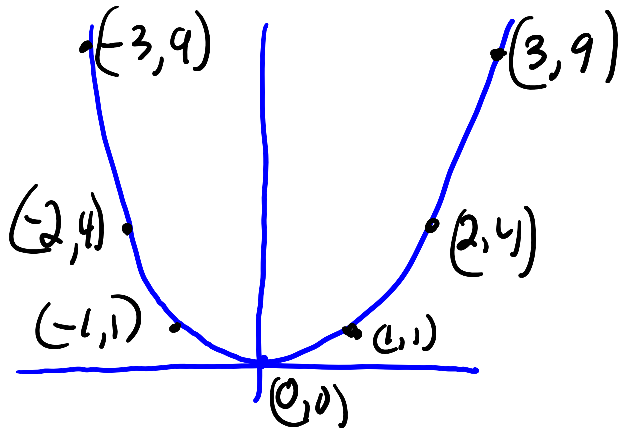
(c) Set up a definite integral that represents the area between the graph of $f(x)$ and the x axis from a to d.

$$USA = \int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

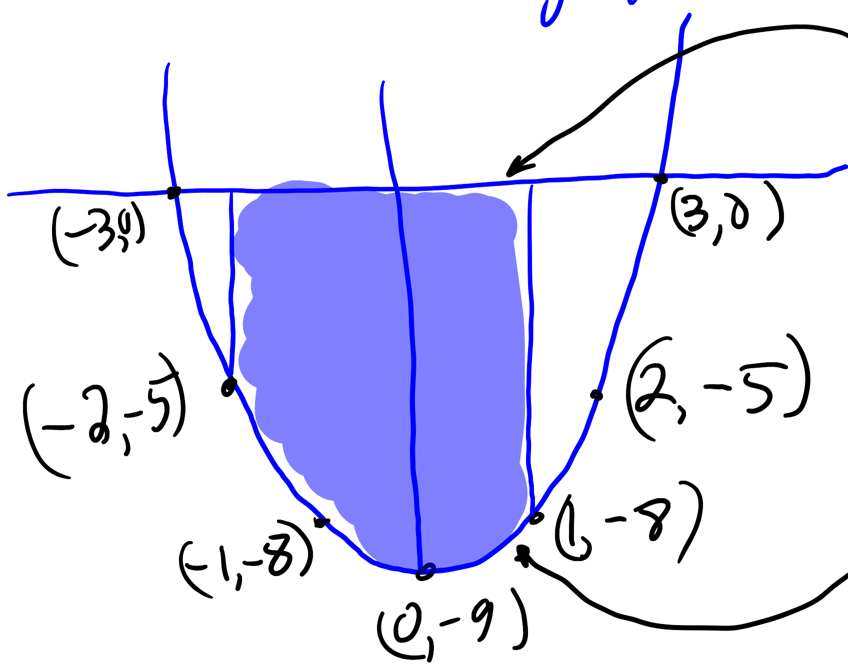
[Example 2](Similar to 6.1#17)

Find the area bounded by the graphs of $y = x^2 - 9$ and $y = 0$ over the interval $-2 \le x \le 1$

graph $y = x^2$ first



$y = x^2 - 9$ will be that graph moved down 9 units



$top(x) = 0$

$bottom(x) = x^2 - 9$

$$\text{USA} = \int_{-2}^{1} \text{top}(x) - \text{bottom}(x)\,dx = \int_{-2}^{1} 0 - (x^2 - 9)\,dx =$$

$$= \int_{-2}^{1} -x^2 + 9\,dx$$

$$\underset{\text{FTC}}{=} \left(\int -x^2 + 9\,dx \right)\Bigg|_{-2}^{1}$$

Indefinite Integral Details

$$\int -x^2 + 9\,dx = -\frac{x^3}{3} + 9x + C$$

$$= \left(-\frac{x^3}{3} + 9x + C \right)\Bigg|_{-2}^{1}$$

$$= \left(-\frac{(1)^3}{3} + 9(1) + \cancel{C} \right) - \left(-\frac{(-2)^3}{3} + 9(-2) + \cancel{C} \right)$$

$$= \left(-\frac{1}{3} + 9 \right) - \left(-\frac{(-8)}{3} - 18 \right)$$

$$= -\frac{1}{3} + 9 - \frac{8}{3} + 18$$

$$= -\frac{9}{3} + 27$$

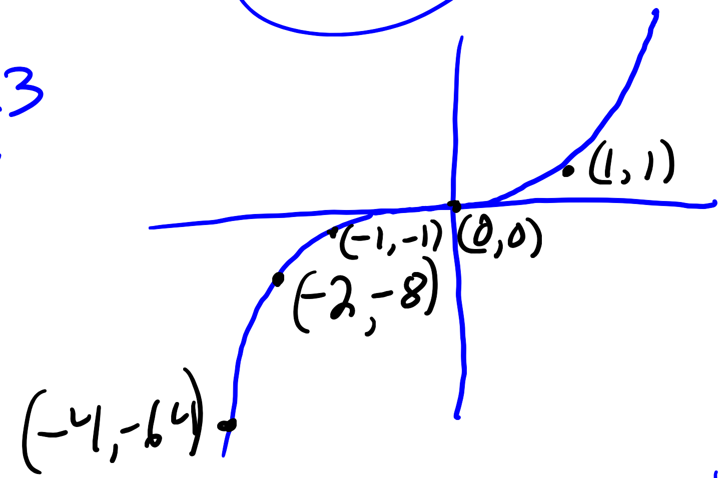
$$= 24$$

[Example 3](Similar to 6.1#21)

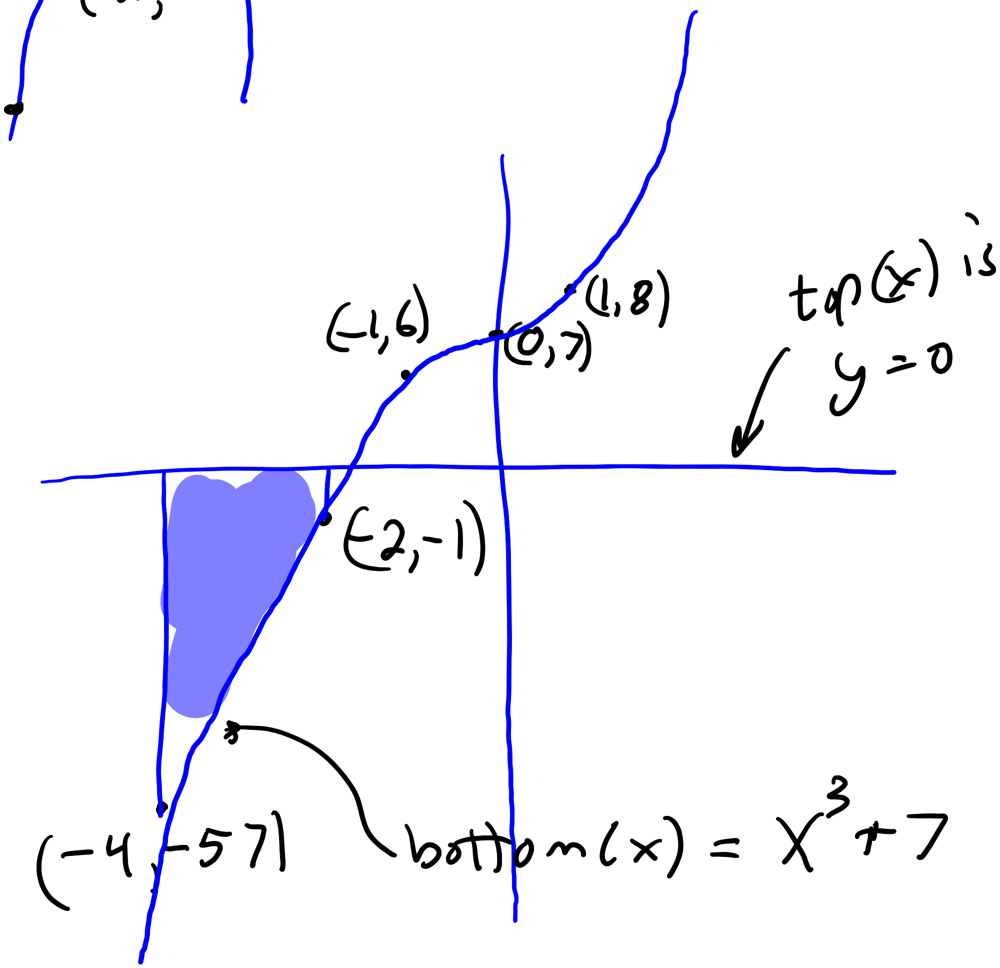
Find the area bounded by the graphs of $y = x^3 + 7$ and $y = 0$ over the interval $-4 \le x \le 2$

Give an exact answer.

Graph $y = x^3$



$y = x^3 + 7$



$$USA = \int_{-4}^{-2} 0 - (x^3+7)\,dx$$

$$= \int_{-4}^{-2} -x^3 - 7\,dx$$

$$\underset{FTC}{=} \left(\underline{\int -x^3 - 7\,dx} \right)\Bigg|_{-4}^{-2}$$

$$= \left(\overline{\frac{-x^4}{4} - 7x + C} \right)\Bigg|_{-4}^{-2}$$

$$= \left(-\frac{(-2)^4}{4} - 7(-2) + \cancel{C} \right) - \left(-\frac{(-4)^4}{4} - 7(-4) + \cancel{C} \right)$$

$$= \left(-\frac{16}{4} + 14 \right) - \left(-\frac{256}{4} + 28 \right)$$

$$= -4 + 14 - (-64 + 28)$$

$$= 10 - (-36)$$

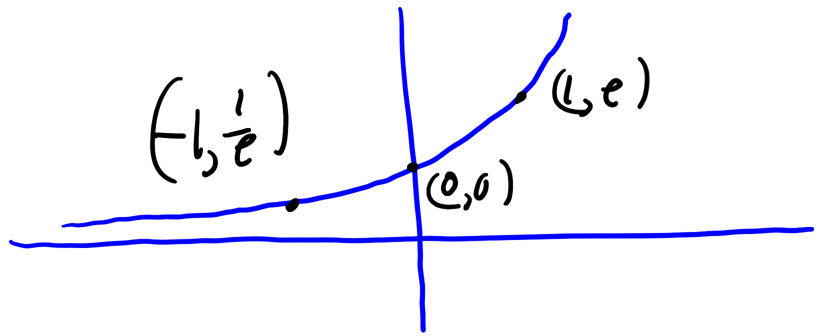
$$= 46$$

[Example 4](Similar to 6.1#23)

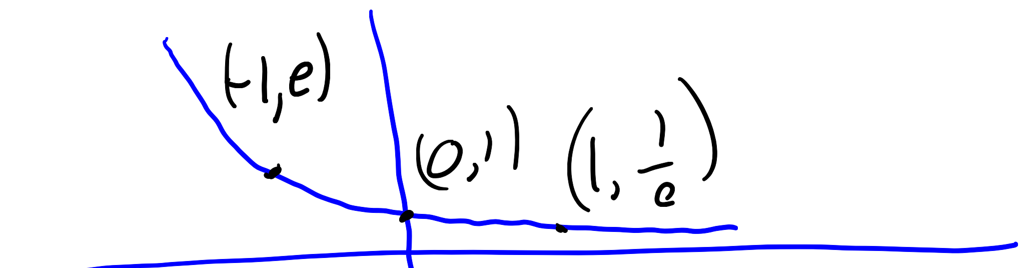
Find the area bounded by the graphs of $y = -e^{(-x)}$ and $y = 0$ over the interval $-1 \le x \le 1$

Give an exact answer and a decimal approximation, rounded to three decimal places.

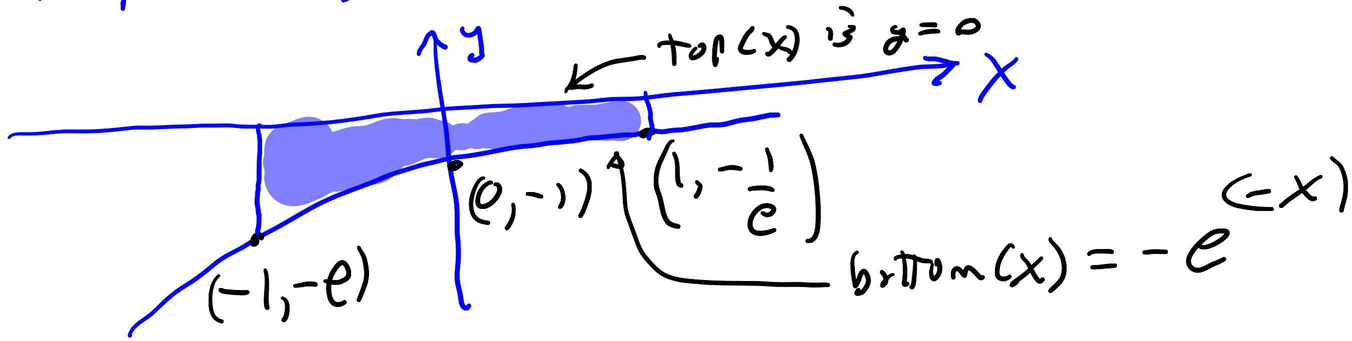
$y = e^{(x)}$



$y = e^{(-x)}$ flip across the y axis



$y = -e^{(-x)}$ flip across the x axis



$$\text{USA} = \int_{-1}^{1} \text{top}(x) - \text{bottom}(x)$$

$$= \int_{-1}^{1} 0 - (-e^{(-x)})\,dx$$

$$= \int_{-1}^{1} e^{(-x)}\,dx$$

FTC
$$= \left(\int e^{(-x)}\,dx \right) \Bigg|_{-1}^{1}$$

$$= \left(-e^{(-x)} + C \right) \Bigg|_{-1}^{1}$$

$$= \left(-e^{(-(1))} + \cancel{C}\right) - \left(-e^{(-(-1))} + \cancel{C}\right)$$

$$= -e^{-1} + e^{1}$$

$$= -\frac{1}{e} + e = e - \frac{1}{e} \approx 2.350$$

exact answer   decimal approximation

$$\boxed{\begin{aligned} \frac{d}{dx} e^{(kx)} &= k e^{(kx)} \\ \int e^{(kx)}\,dx &= \frac{e^{(kx)}}{k} + C \end{aligned}}$$

Indefinite Integral Details

$$\int e^{-x}\,dx = \int e^{(-1\cdot x)}\,dx$$

$k = -1$

$$= \frac{e^{(-1\cdot x)}}{(-1)} + C$$

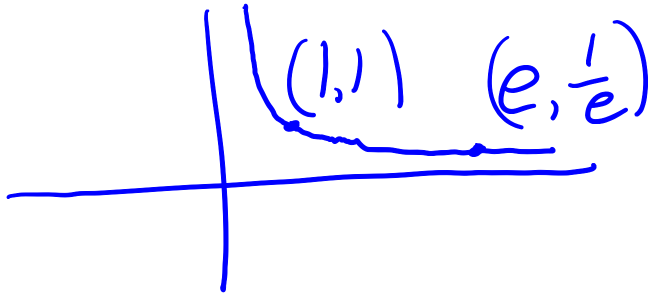
$$= -e^{(-x)} + C$$

[Example 4](Similar to 6.1#25)

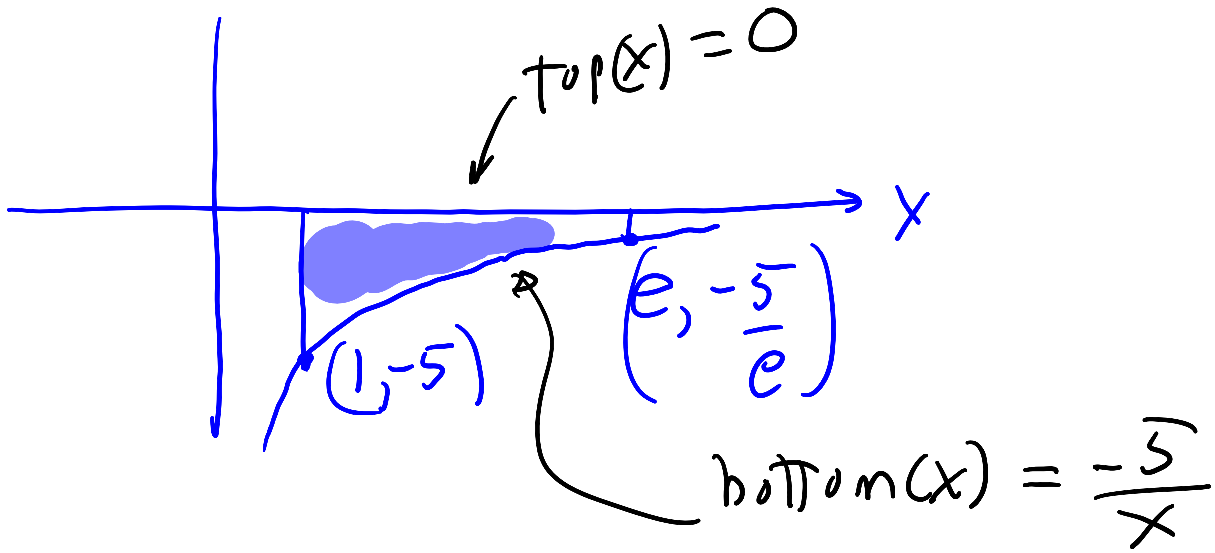
Find the area bounded by the graphs of $y = -\frac{5}{x}$ and $y = 0$ over the interval $-1 \leq x \leq e$

Give an exact answer and a decimal approximation, rounded to three decimal places.

$y = \frac{1}{x}$



$y = -\frac{5}{x}$



$$USA = \int_1^e top(x) - bottom(x)\,dx$$

$$= \int_1^e 0 - \left(-\frac{5}{x}\right) dx$$

$$= \int_1^e \frac{5}{x}\,dx$$

$$\underset{FTC}{=} \left(\int \frac{5}{x}\,dx\right)\Bigg|_1^e$$

Indefinite Integral Details

$$\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx$$

$$= 5\ln(|x|) + c$$

$$= \left(5\ln(|x|) + c\right)\Big|_1^e$$

$$= \left(5\ln(|e|) + \cancel{c}\right) - \left(5\ln(|1|) + \cancel{c}\right)$$

$$= 5\,\underbrace{\ln(e)}_{1} - 5\,\underbrace{\ln(1)}_{0}$$

$$= 5$$

[Example 5](Similar to 6.1#27)

Find the area bounded by the graphs of $y = \sqrt{49 - x^2}$ and $y = 0$ over the interval $0 \le x \le 7$

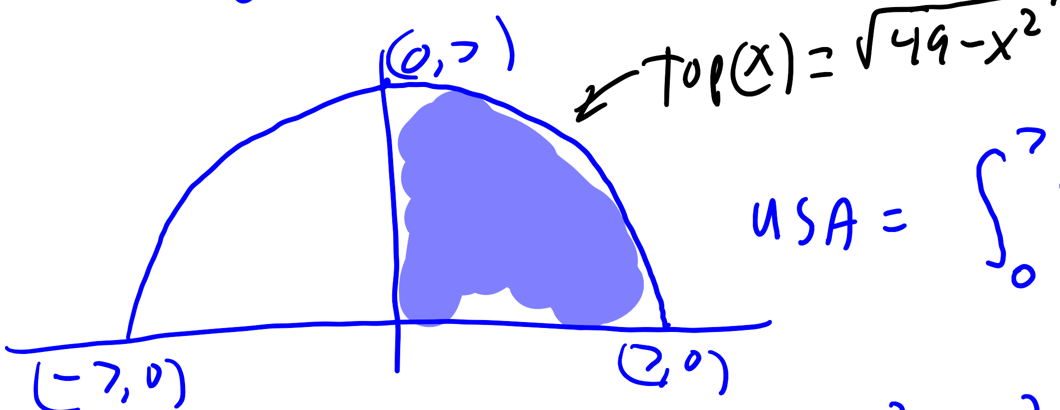
Give an exact answer and a decimal approximation, rounded to three decimal places.

notice $x^2 + y^2 = 49$ is a circle, $r = 7$, centered at $(0,0)$

$$y^2 = 49 - x^2$$

$$y = \pm\sqrt{49 - x^2}$$

So $y = \sqrt{49 - x^2}$ will be the top half of the circle



$$USA = \int_0^7 \sqrt{49 - x^2}\,dx = ??$$

We can use geometry! $USA = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi (7)^2 = \frac{1}{4}\pi \cdot 49$

$$= \frac{49\pi}{4} \approx 38.485$$

exact approximate