Subject for this video:

The Area Between Curves

Reading:

- **General:** Section 6.1 The Area Between Curves
- **More Specifically:** Pages 390 - 391 Examples 3,4,5

Homework: H83: Area between two curves

- Barnett 6.1#37,53,55
- Briggs & Cochran 6.2#9,42,46

Recall the theorem introduced in the previous video.

Theorem about the Area between Curves for a Simple Region.

If $top(x)$ and $bottom(x)$ are continuous functions and $bottom(x) \leq top(x)$ on the interval $[a, b]$, then the region bounded by the four curves

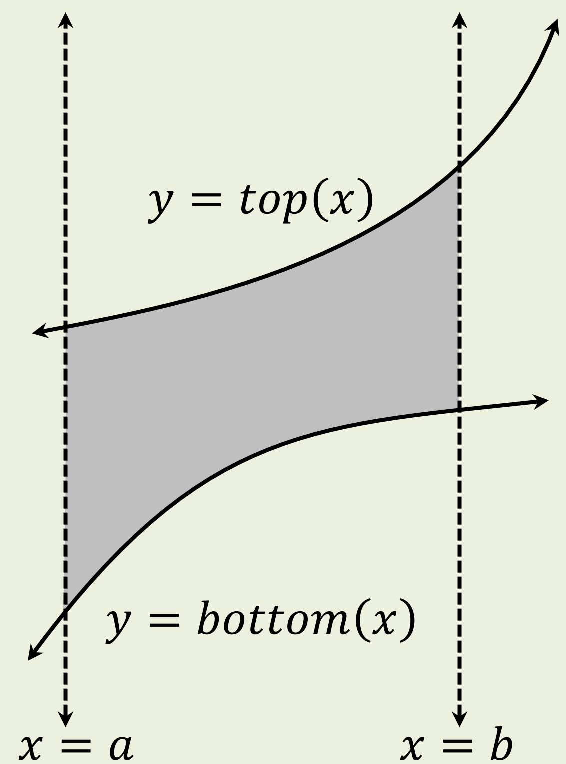
$$y = top(x)$$

$$y = bottom(x)$$

$$x = a$$

$$x = b$$

is what we in this course will call a ***simple region***.



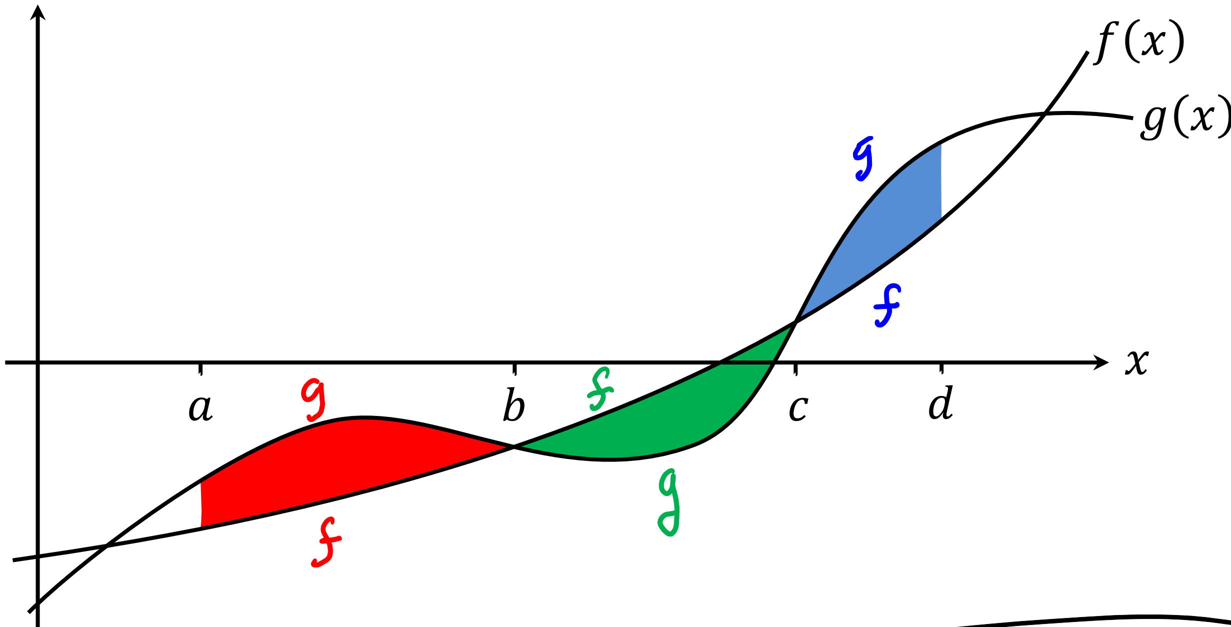
The ***Area Between Curves*** (*unsigned* area) for this region is given by the definite integral

$$USA = \int_a^b top(x) - bottom(x)dx$$

In the previous video, we applied this theorem to find the area between curves in examples where one of the curves was the x axis. That familiarized us with the method of setting up the integrals while keeping the calculations simple.

In this video, we will study examples where neither of the curves involved is the x axis. That will make the calculations harder.

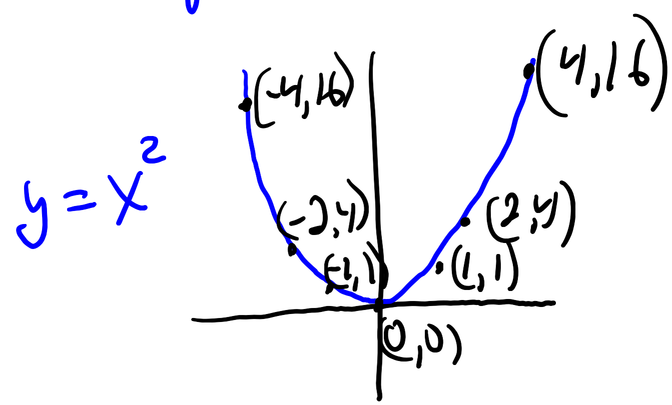
[Example 1](Similar to 6.1#37) Set up a definite integral calculation that computes the area between the graphs of $f(x)$ and $g(x)$ for $a \leq x \leq d$.

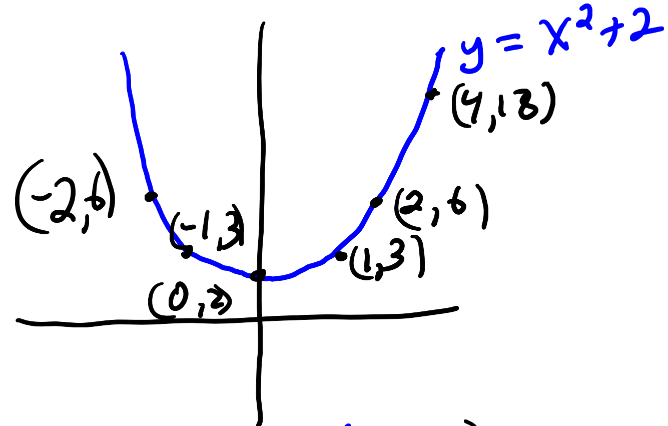


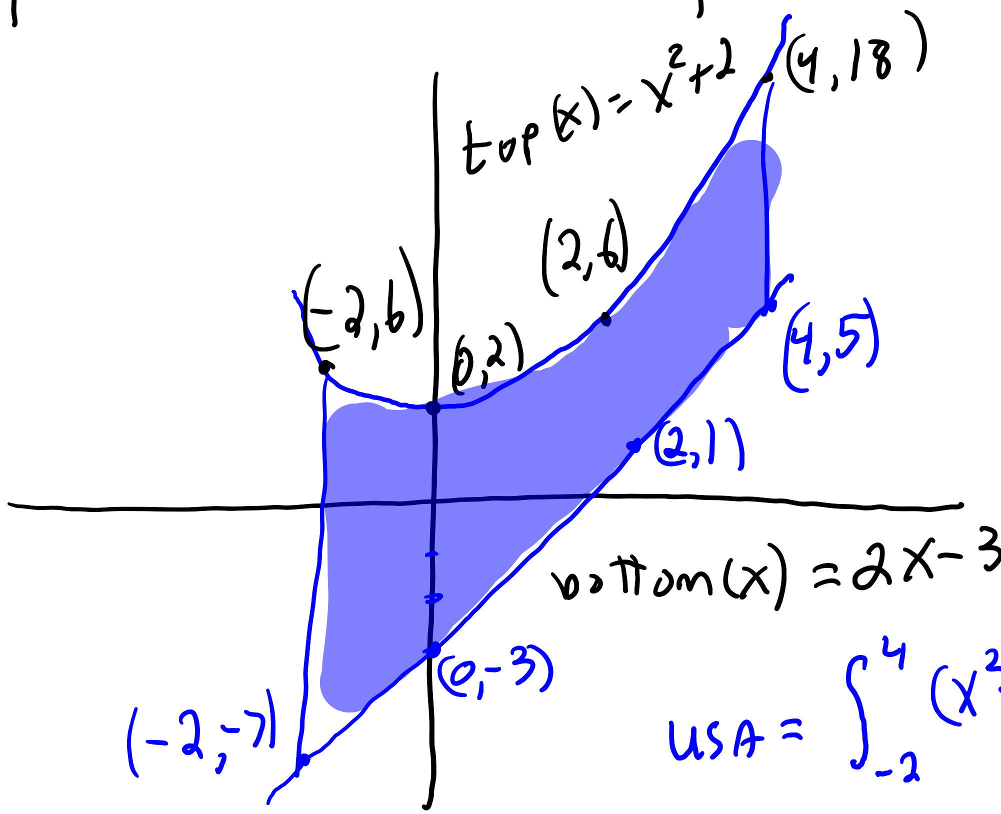
$$\text{USA} = \int_a^b g(x) - f(x)\,dx + \int_b^c f(x) - g(x)\,dx + \int_c^d g(x) - f(x)\,dx$$

[Example 2](similar to 6.1#53) Find the area bounded by the graphs of the equations $y = x^2 + 2$ and $y = 2x - 3$ over the interval $-2 \le x \le 4$. Give an exact, simplified answer.

Must graph the equations in order to understand the region







top(x) - bottom(x)

$$USA = \int_{-2}^{4} (x^2+2) - (2x-3)\, dx$$

$$USA = \int_{-2}^{4} \underbrace{(x^2+2)-(2x-3)}\,dx$$

simplify integrand before integrating!

$$= \int_{-2}^{4} \overbrace{x^2-2x+5}\,dx$$

$$\underset{FTC}{=} \left(\int x^2-2x+5\,dx \right)\Big|_{-2}^{4}$$

Indefinite Integral Details

rewrite the integrand

$$f(x) = x^2-2x+5$$

$$= x^2-2\cdot x^1+5\cdot x^0$$

$$F(x) = \int x^2-2x^1+5x^0\,dx$$

$$= \frac{x^{2+1}}{2+1} - \frac{2x^{1+1}}{1+1} + \frac{5x^{0+1}}{0+1} + C$$

$$= \frac{x^3}{3} - x^2 + 5x + C$$

$$= \left(\frac{x^3}{3} - x^2 + 5x + C\right)\Big|_{-2}^{4}$$

$$= \left(\frac{(4)^3}{3} - (4)^2 + 5(4) + \cancel{C}\right) - \left(\frac{(-2)^3}{3} - (-2)^2 + 5(-2) + \cancel{C}\right)$$

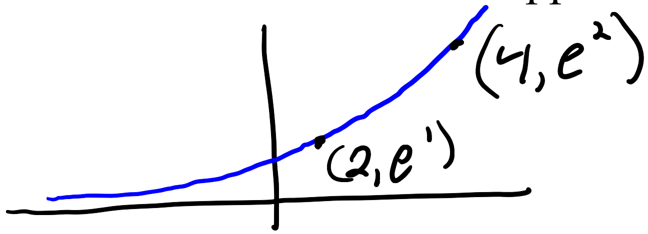
$$= \left(\frac{64}{3} - 16 + 20\right) - \left(\frac{-8}{3} - 4 - 10\right)$$

$$= \frac{64}{3} + 4 + \frac{8}{3} + 14 = \frac{72}{3} + 18 = 24 + 18 = 42$$

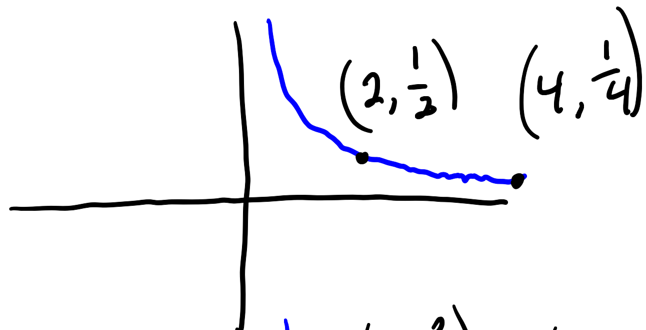
[Example 3](similar to 6.1#55) Find the area bounded by the graphs of the equations $y = e^{(0.5x)}$ and $y = -\frac{3}{x}$ over the interval $2 \le x \le 4$.

Give an exact, simplified answer and a decimal approximation rounded to 3 decimal places.

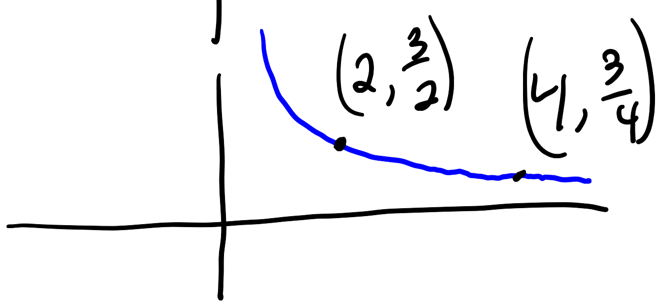
$y = e^{(.5x)}$



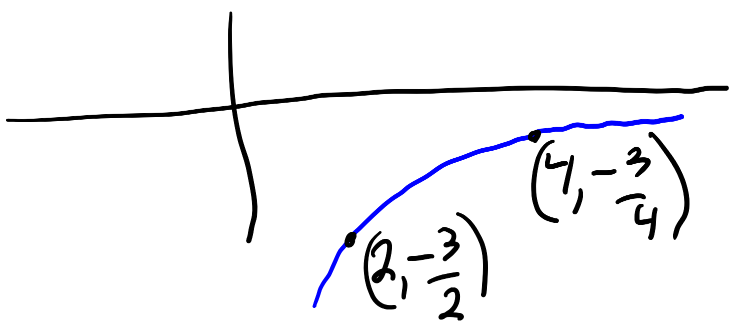
$y = \frac{1}{x}$

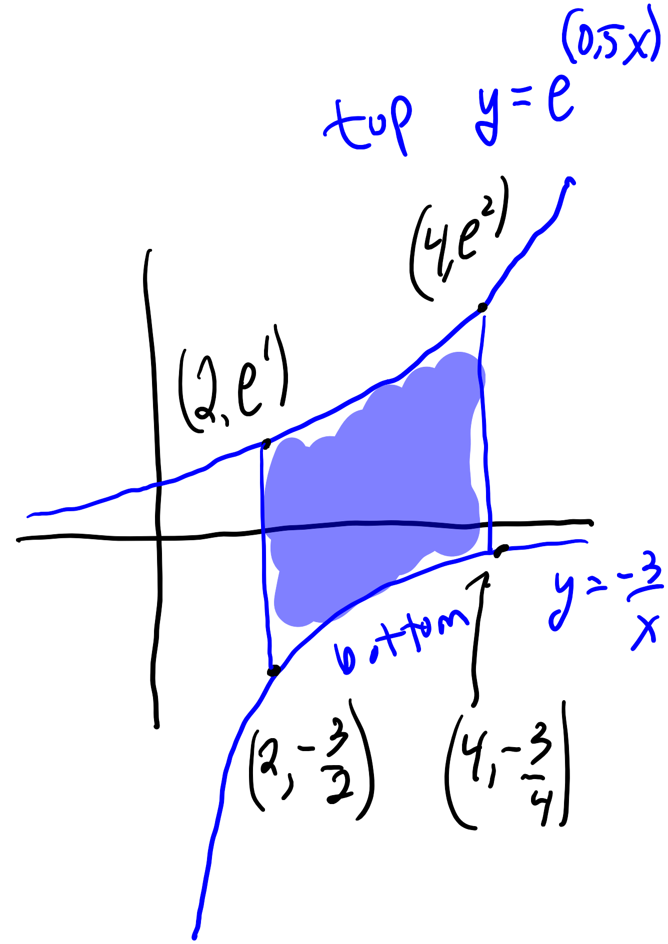


$y = \frac{3}{x}$



$y = -\frac{3}{x}$





$$USA = \int_2^4 \underbrace{e^{(0.5x)} - \left(-\frac{3}{x}\right)}_{\text{Simplify before integrating}} dx$$

$$= \int_2^4 e^{(0.5x)} + \frac{3}{x}\, dx$$

$$\overset{FTC}{=} \left(\int e^{(0.5x)} + \frac{3}{x}\, dx \right)\Bigg|_2^4$$

Indefinite Integral Details

$$F(x) = \int e^{(0.5x)} + \frac{3}{x}\, dx$$

$$= \int e^{(0.5x)} dx + 3\int \frac{1}{x}\, dx$$

$$= \frac{e^{(0.5x)}}{0.5} + 3\ln(|x|) + C$$

$$= 2e^{(0.5x)} + 3\ln(|x|) + C$$

$$= \left(2e^{(0.5x)} + 3\ln(|x|) + C\right)\Big|_2^4$$

$$= \left(2e^{(0.5(4))} + 3\ln(|4|) + \cancel{C}\right) - \left(2e^{(0.5(2))} + 3\ln(|2|) + \cancel{C}\right)$$

$$= \left(2e^2 + 3\ln(4)\right) - \left(2e^1 + 3\ln(2)\right)$$

$$= 2e^2 - 2e + 3\ln(4) - 3\ln(2) = 2e^2 - 2e + 3\ln\left(\frac{4}{2}\right)$$

$$= 2e^2 - 2e + 3\ln(2) \approx 11.421$$

exact answer

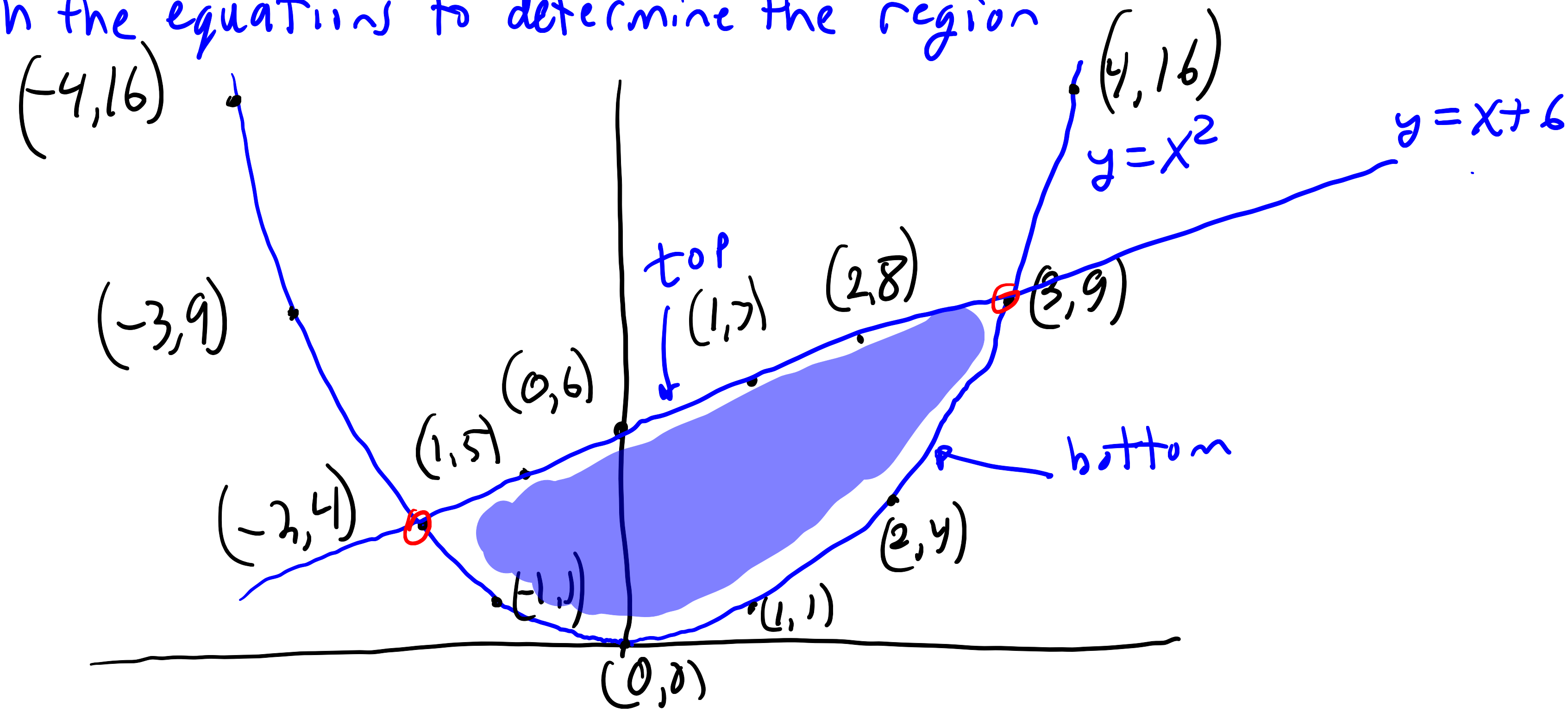
decimal approximation

[Example 4](similar to Briggs & Cochran 6.2#9)

Find the area bounded by the graphs of the equations $y = x^2$ and $y = x + 6$.

Give an exact, simplified answer and a decimal approximation rounded to 3 decimal places.

Graph the equations to determine the region



$$USA = \int_{-2}^{3} (x+6) - x^2 \, dx = \int_{-2}^{3} -x^2 + x + 6 \, dx$$

$$\text{USA} = \int_{-2}^{3} -x^2+x+6\,dx$$

$$\underset{\text{FTC}}{=} \left(\int -x^2+x+6\,dx\right)\Bigg|_{-2}^{3}$$

$$= \left(-\frac{x^3}{3}+\frac{x^2}{2}+6x+C\right)\Bigg|_{-2}^{3}$$

$$= \left(-\frac{(3)^3}{3}+\frac{(3)^2}{2}+6(3)+\cancel{C}\right)-\left(-\frac{(-2)^3}{3}+\frac{(-2)^2}{2}+6(-2)+\cancel{C}\right)$$

$$= \left(-\frac{27}{3}+\frac{9}{2}+18\right)-\left(-\frac{(-8)}{3}+\frac{4}{2}-12\right)$$

$$= \left(-9+\frac{9}{2}+18\right)-\left(\frac{8}{3}+2-12\right) = \left(9+\frac{9}{2}\right)-\left(\frac{8}{3}-10\right)$$

$$= 19+\frac{9}{2}-\frac{8}{3} = 19+\frac{27}{6}-\frac{16}{6} = 19+\frac{11}{6} = \frac{114}{6}+\frac{11}{6}$$

$$= \frac{125}{6} \approx 20.833$$

exact approximate

Indefinite Integral Details

Rewrite the integrand

$$f(x) = -x^2+x^1+6\cdot x^0$$

Integrate

$$F(x) = \int -x^2+x^1+6x^0\,dx$$

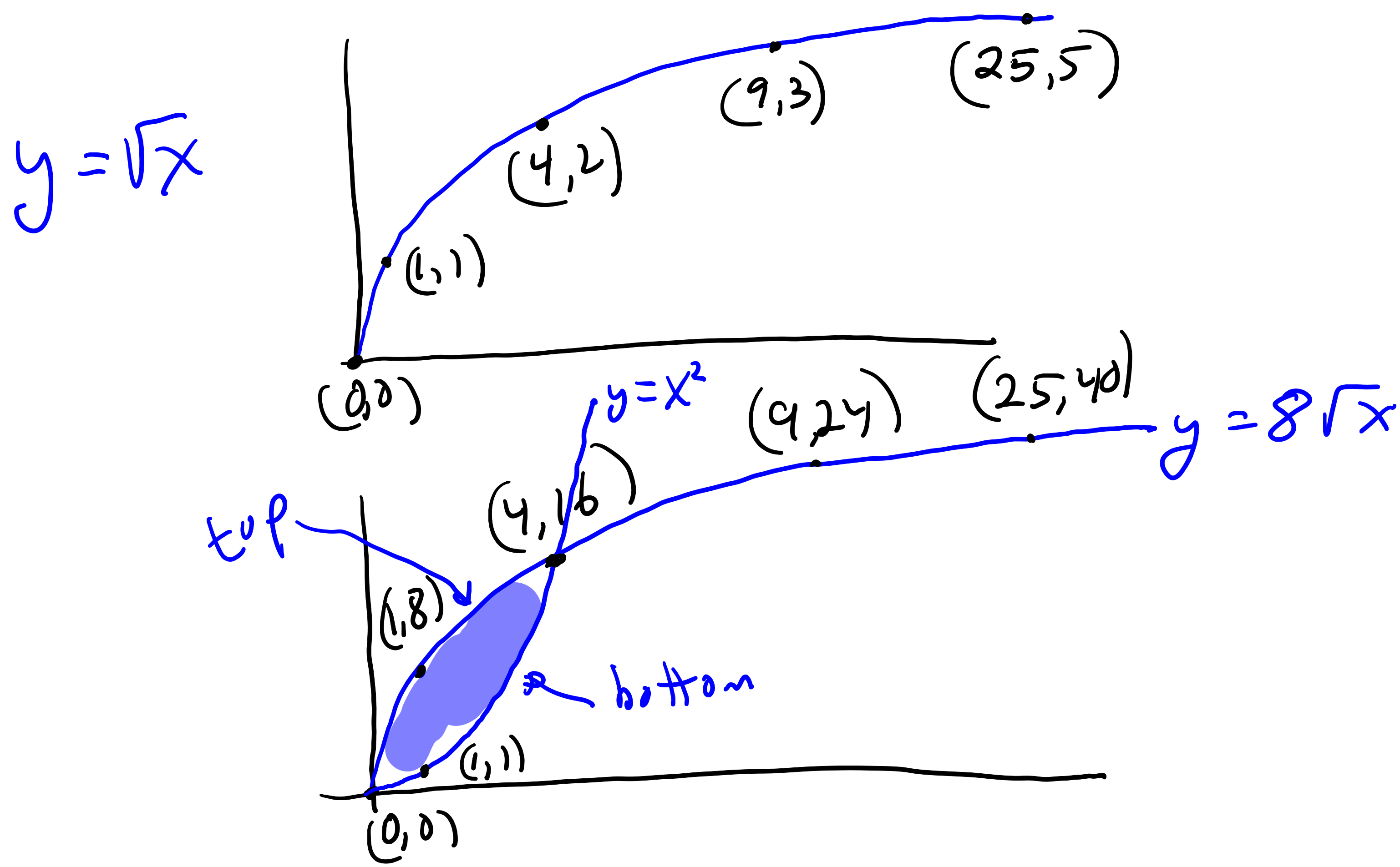
$$= -\frac{x^3}{3}+\frac{x^2}{2}+\frac{6x^1}{1}+C$$

$$= -\frac{x^3}{3}+\frac{x^2}{2}+6x+C$$

[Example 5](similar to Briggs & Cochran 6.2#42)

Find the area bounded by the graphs of the equations $y = x^2$ and $y = 8\sqrt{x}$.

Give an exact, simplified answer and a decimal approximation rounded to 3 decimal places.



$$USA = \int_0^4 8\sqrt{x} - x^2\,dx$$

$$\text{US } A = \int_0^4 8\sqrt{x} - x^2\,dx$$

$$\underset{\text{FTC}}{=} \left(\underbrace{\int 8\sqrt{x} - x^2\,dx} \right)\Bigg|_0^4$$

$$= \left(\frac{16}{3}x^{\frac{3}{2}} - \frac{x^3}{3} + C \right)\Bigg|_0^4$$

$$= \left(\frac{16}{3}(4)^{3/2} - \frac{(4)^3}{3} + \cancel{C} \right) - \left(\frac{16}{3}(0)^{3/2} - \frac{(0)^3}{3} + \cancel{C} \right)$$

$$= \frac{16}{3}\left((4)^{1/2}\right)^3 - \frac{64}{3} = \frac{16(2)^3}{3} - \frac{64}{3}$$

$$= \frac{16(8)}{3} - \frac{64}{3} = \frac{128}{3} - \frac{64}{3}$$

$$= \frac{64}{3}$$ exact answer

$$\approx 21.333$$ decimal approximation.

Indefinite Integral Details

rewrite the integrand

$$f(x) = 8\sqrt{x} - x^2 = 8x^{\frac{1}{2}} - x^2$$

Integrate

$$F(x) = \int 8x^{\frac{1}{2}} - x^2\,dx$$

$$= \frac{8x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{2+1}}{2+1} + C$$

$$= \frac{8x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} + C$$

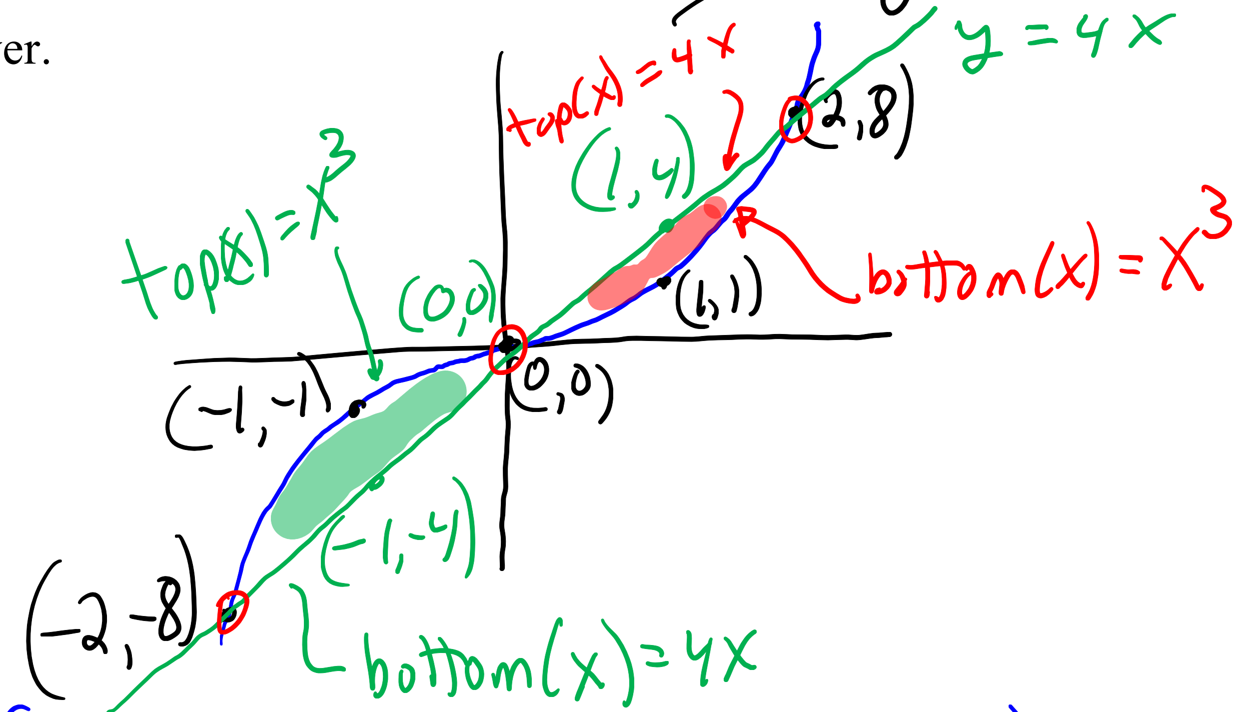
$$= 8\left(\frac{2}{3}\right)x^{3/2} - \frac{x^3}{3} + C$$

$$= \frac{16}{3}x^{3/2} - \frac{x^3}{3} + C$$

[Example 6](similar to Briggs & Cochran 6.2#46)

Find the area bounded by the graphs of the equations $y = x^3$ and $y = \cancel{8x}$. $y = 4x$

Give an exact, simplified answer.



We need one definite integral for each simple region

$$USA = \int_{-2}^{0} x^3 - 4x \, dx + \int_{0}^{2} 4x - x^3 \, dx$$

$$USA = \int_{-2}^{0} x^3 - 4x\,dx + \int_{0}^{2} 4x - x^3\,dx$$

$$\overset{FTC}{=} \left(\int x^3 - 4x\,dx\right)\Bigg|_{-2}^{0} + \left(\int 4x - x^3\,dx\right)\Bigg|_{0}^{2}$$

$$= \left(\frac{x^4}{4} - 2x^2 + C\right)\Bigg|_{-2}^{0} + \left(2x^2 - \frac{x^4}{4} + D\right)\Bigg|_{0}^{2}$$

$$= \left[\left(\frac{(0)^4}{4} - 2(0)^2 + \cancel{C}\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2 + \cancel{C}\right)\right] + \left[\left(2(2^2) - \frac{(2)^4}{4} + \cancel{D}\right) - \left(2(0)^2 - \frac{(0)^4}{4} + \cancel{D}\right)\right]$$

$$= -\left(\frac{16}{4} - 2(4)\right) + \left(2(4) - \frac{16}{4}\right)$$

$$= -4 + 8 + 8 - 4$$

$$= 8$$