Subject for this video:

Total Change Problems as Area Problems

Reading:

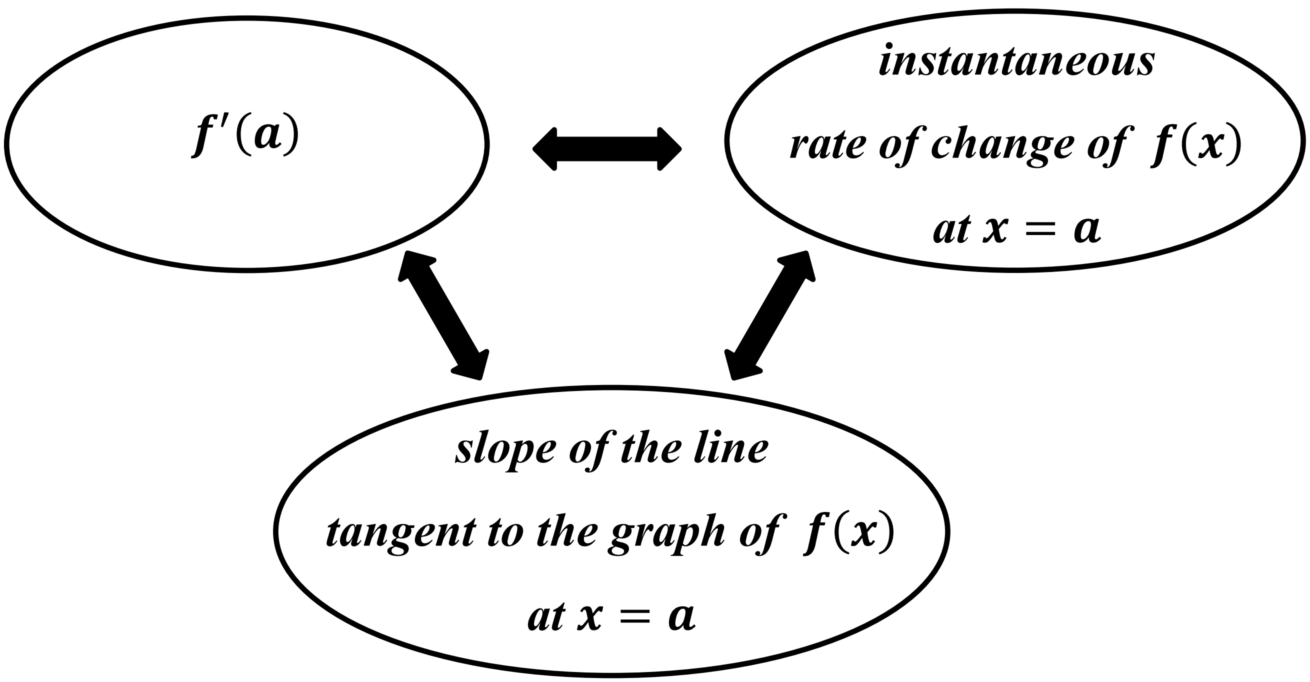
- **General:** Section 6.1 The Area Between Curves
- **More Specifically:** Pages 390 - 391 Examples 3,4,5

Homework: H84: Applications of the Area Between Two Curves: Total Change (6.1#89,91)

Recall that the most important concept of the second month of the course:

> **Three Equal Quantities Related to *Slope***
>
> **words:** the ***instantaneous rate of change of*** $\boldsymbol{f(x)}$ ***at*** $\boldsymbol{x = a}$
>
> **words:** the ***slope of the line tangent to the graph of*** $\boldsymbol{f(x)}$ ***at*** $\boldsymbol{x = a}$
>
> **words:** the ***derivative of*** $\boldsymbol{f(x)}$ ***at*** $\boldsymbol{x = a}$
>
> **symbol:** $\boldsymbol{f'(a)}$

The concept was visualized with this diagram:



Three Equal Quantities Related to Slope

(the most important concept of the second month of the course)

It is useful now to identify three important equal quantities related to *area*.

Recall that the *definite integral* represents a *signed area*:

The *Definite Integral and Signed Area*

Symbol: $\int_a^b f(x)dx$

Spoken: The ***definite integral*** of $f(x)$ from a to b.

Informal meaning, in terms of the graph: The *signed area* of the region between the graph of $f(x)$ and the x axis on the interval $[a, b]$.

And recall the relationship between definite integral and antiderivatives, articulated in the *Fundamental Theorem of Calculus*.

The *Fundamental Theorem of Calculus* (*FTC*)

(the relationship between ***definite integrals*** and ***antiderivatives***)

If $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

And recall that the definite integral can be used to find the total change of a function $F(x)$ from $x = a$ to $x = b$, if the derivative $F'(x)$ is known.

Definition of *Total Change Problems*

The Problem:

- **Given:** the rate of change of some quantity, $F'(x)$, and two numbers a, b with $a \leq b$,
- **Find:** the change in the quantity, $\Delta F = F(b) - F(a)$

Solution to the Problem: Use the Fundamental Theorem of Calculus

$$\Delta F = F(b) - F(a) \underset{FTC}{=} \int_a^b F'(x)$$

(This is simply a re-packaging of the Fundamental Theorem of Calculus.)

It is useful to identify three equal quantities:

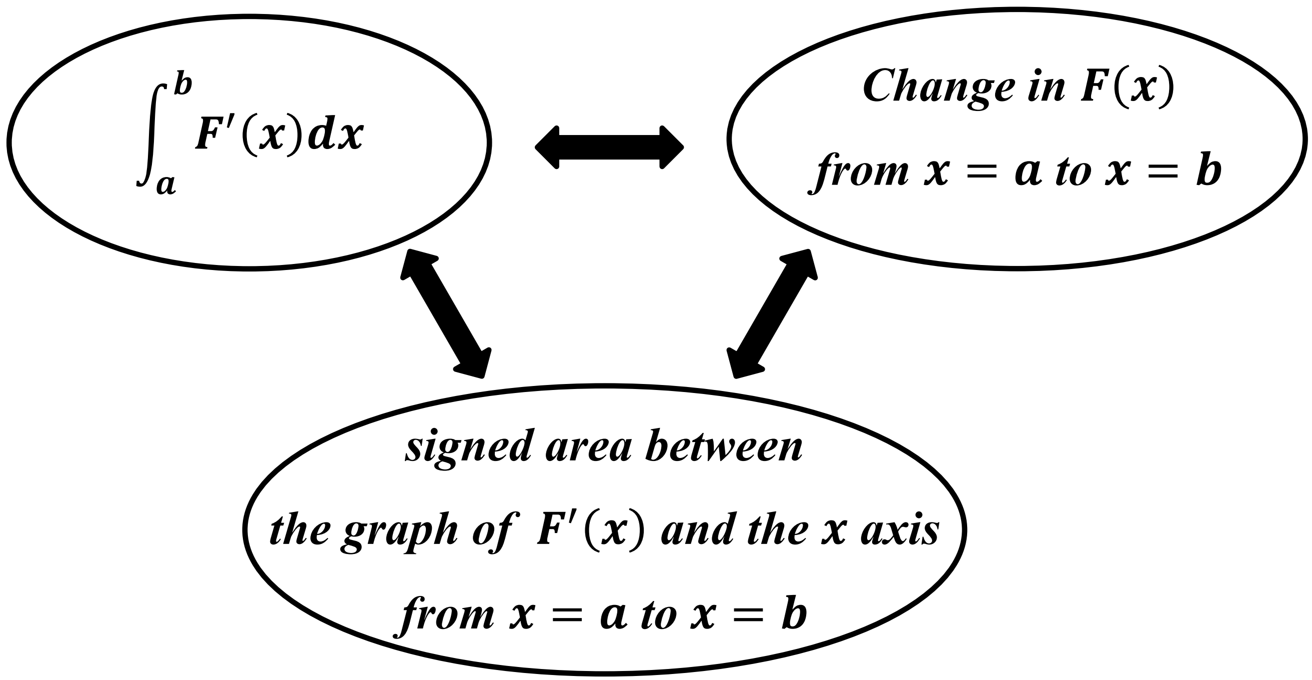
Three Equal Quantities Related to *Area*

words: the ***change in $F(x)$ from $x = a$ to $x = b$***

words: the ***signed area between the graph of $F'(x)$ and the x axis from $x = a$ to $x = b$***

words and symbol: the ***definite integral of $F'(x)$ from a to b,*** $\int_a^b F'(x)dx$

And it is helpful to visualize the relationship with a diagram:



Three Equal Quantities Related to Area

(An important concept in the fourth month of the course)

In Homework H80 Total Change Problems, you were asked to find the *total change in* $F(x)$ *from* $x = a$ *to* $x = b$ when the the derivative $F'(x)$ was given. The point of those exercises was that you had to make the connection that the change in $F(x)$ is found by integrating $F'(x)$.

In Homework H84, the subject of this video, you are asked to compute areas between graphs of $F'(t)$ and the t axis over a time interval $a \leq t \leq b$, and then to *interpret* the result. The point of these exercises is that you must first make the connection that the area is computed by integrating $F'(t)$, and then make the connection that what that area *represents* is the *change in* $F(t)$ *from* $t = a$ *to* $t = b$.

[Example 1] Bacteria Growth Rate and Change in Weight

A bacteria culture is growing at a rate

$$W'(t) = .6e^{(.2t)}\ grams\ per\ hour$$

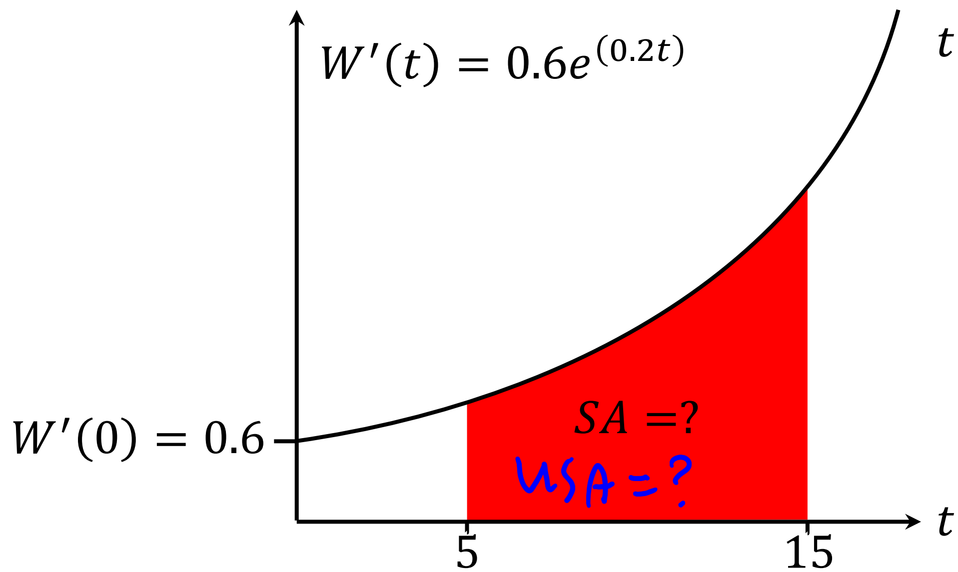
(a) Find the area between the graph of $W'(t)$ and the t axis over the interval [5,15].

(Give an exact answer and a decimal approximation)

Solution:

Since $e^{anything} > 0$, we know that

- The curve $W'(t) = .6e^{(.2t)}$ will be on top.
- the curve $y = 0$ will be on bottom.



So the area will be computed by the definite integral.

$$USA = \int_5^{15} top(t) - bottom(t)dt = \int_5^{15} W'(t) - 0dt = \int_5^{15} W'(t)dt$$

The integral is computed on the next page.

$$USA = \int_5^{15} 0.6e^{(0.2t)}dt$$

$$\underset{FTC}{=} \left(\int 0.6e^{(0.2t)}dt\right)\Big|_5^{15}$$

$$= \left(3e^{(0.2t)} + C\right)\Big|_5^{15}$$

$$= \left(3e^{(0.2(15))} + C\right) - \left(3e^{(0.2(5))} + C\right)$$

$$= 3e^{(3)} - 3e^{(1)}$$

$$= 3e^3 - 3e \quad \text{exact}$$

$$\approx 52.1 \quad \text{approximation}$$

Indefinite Integral Details

$$W(t) = \int W'(t)dt$$

$$= \int 0.6e^{(0.2t)}dt$$

$$= 0.6\int e^{(0.2t)}dt + C$$

$$= 0.6\left(\frac{e^{(0.2t)}}{0.2}\right) + C$$

$$= 3e^{(0.2t)} + C$$

$$\int e^{(kx)}dx = \frac{e^{(kx)}}{k} + C$$

$$\frac{d}{dx} e^{(kx)} = k e^{(kx)}$$

(b) Interpret the results of part **(a)**.

Solution:

The result of part **(a)** is an abstract mathematical result. To *interpret that result* means to explain what that result tells us about the bacteria culture.

In part **(a)**, we found the unsigned area (USA) using the definite integral

$$USA = \int_{5}^{15} W'(t)dt$$

But we realize that the same definite integral computes the signed area

$$SA = \int_{5}^{15} W'(t)dt$$

By the Fundamental Theorem of Calculus, the value of this integral equals the change in weight of the culture from $t = 5$ hours to $t = 15$ hours.

$$\int_{5}^{15} W'(x) \underset{FTC}{=} W(15) - W(5)$$

That is, the change in weight of the culture from $t = 5$ hours to $t = 15$ hours is roughly 52.1 grams.

That is the "interpretation"

Remark: We have done this same problem before. In the video for Homework H80 (discussing concepts from Section 5.5) the question was posed a little differently, as an example of a total change problem:

From the Video for Homework H80

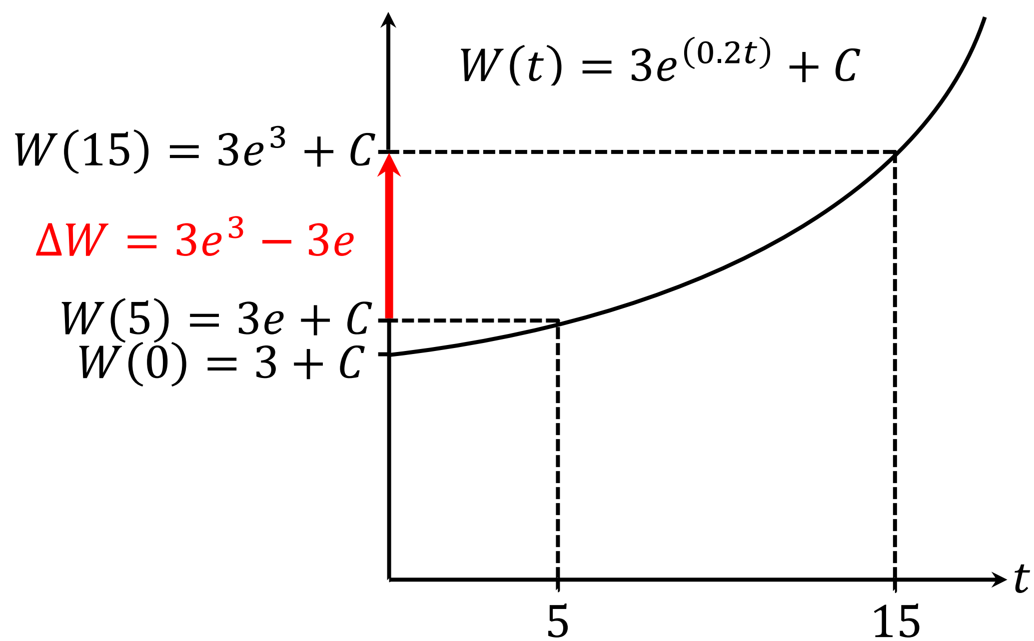
[Example 2] (similar to 5.5#89) Bacteria Growth Rate and Change in Weight

A bacteria culture is growing at a rate

$$W'(t) = .6e^{.2t} \; grams\ per\ hour$$

How much does the weight of the culture change from $t = 5$ hours to $t = 15$ hours?

In that example, we illustrated the change in weight with the graph shown at right.



[End of Example 1]

[Example 2] A training course estimates that its students learn skills at the rate

$$S'(t) = \frac{10}{t+1} \text{ for } 0 \leq t \leq 8$$

where t is the time in hours spent in the training course and $S'(t)$ is the rate at which skills are being learned at time t (in units of skills per hour).

(a) Find the area between the graph of $S'(t)$ and the t axis over the interval $[1,3]$.

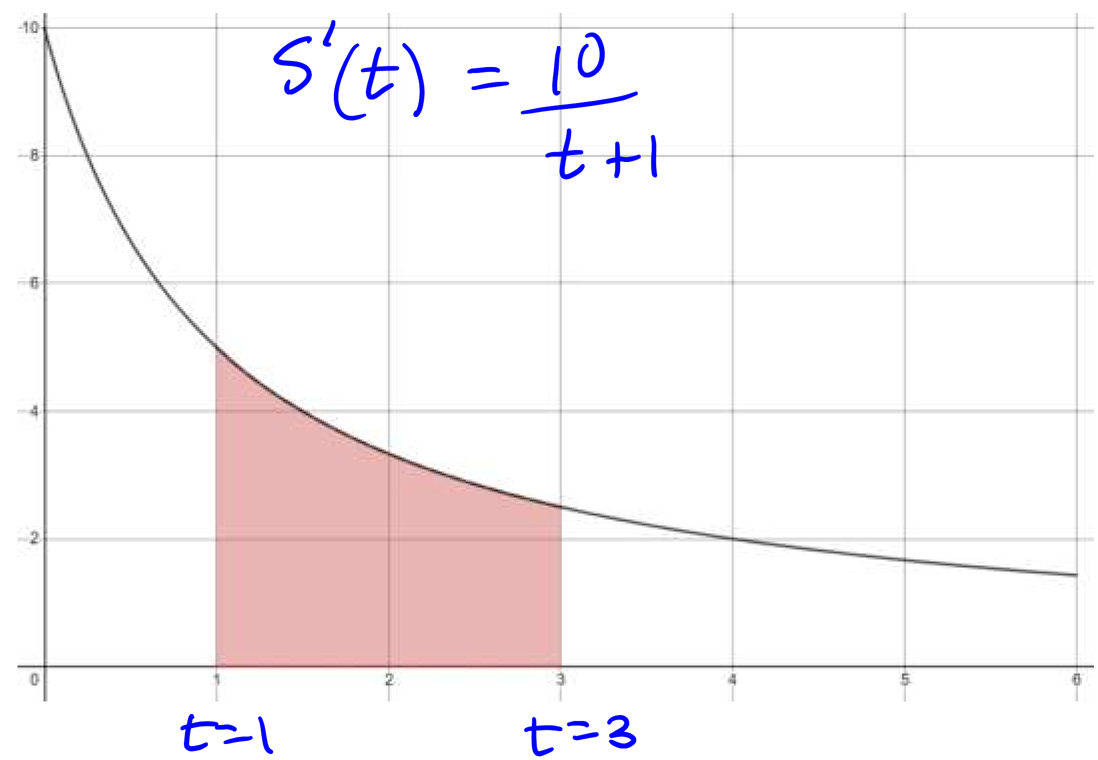
Give an exact answer in symbols and a decimal approximation, rounded to the nearest integer.

Solution: Since values of

$$S'(t) = \frac{10}{t+1}$$

will be positive for $0 \leq t \leq 8$, we know

- The curve $S'(t)$ will be on top.
- the curve $y = 0$ will be on bottom.



So the area will be found by the definite integral.

$$USA = \int_1^3 top(t) - bottom(t)dt = \int_1^3 S'(t) - 0dt = \int_1^3 S'(t)dt = \int_1^3 \frac{10}{t+1}dt$$

Definite Integral Computation

$$USA = \int_1^3 \frac{10}{t+1}dt$$

$$\underset{FTC}{=} \left(\int \frac{10}{t+1}dt\right)\Bigg|_1^3$$

$$= (10\ln(|t+1|))|_1^3 \quad \text{(See Indefinite Integral Details on next page)}$$

+C

$$= (10\ln(|(3)+1|) + C) - (10\ln(|(1)+1|) + C)$$

$$= (10\ln(4)) - (10\ln(2))$$

$$= 10\left(\ln\left(\frac{4}{2}\right)\right)$$

$$= 10\ln(2)$$ exact

$$\approx 7$$ approximate

Indefinite Integral Details (Using Substitution)

$$S(t) = \int S'(t)dt = \int \frac{10}{t+1}dt$$

Step **1**: $u = t + 1$

Step **2**: $u' = 1$ so $dt = du$

$dt = \frac{1}{u'}du$

Step **3**: $\int \frac{10}{t+1}dt = \int \frac{10}{u}du = 10\int \frac{1}{u}du$

substitute simplify

Step **4**: $10\int \frac{1}{u}du = 10\ln(|u|) + C$

Step **5**: $S(t) = \int \frac{10}{t+1}dt = 10\ln(|t+1|) + C$

(b) Interpret the results of part **(a)**.

Solution: The result of part **(a)** is an abstract mathematical result, the value of an area, computed by finding a *definite integral*. To *interpret that result* means to explain what that result tells us about learning skills. It is helpful to consider first what the result of the *indefinite integral* means, and to simplify it further.

The indefinite integral result is the following function form:

$$S(t) = \int \frac{10}{t+1} dt = 10 \ln(|t+1|) + C$$

Two observations will enable us to simplify this function form and get an actual function.

- Note that during the time interval $0 \leq t \leq 6$, the value of $t+1$ will always be positive. Therefore, the expression $|t+1|$ can be simplified to $(t+1)$, and $S(t)$ becomes

$$S(t) = 10 \ln(t+1) + C$$

- Since the value of $S'(t)$ is the rate at which a student learns skills at time t hours, it is also the rate of change of the number of skills that the student knows. That is, the value of the function $S(t)$ is the number of skills that the student knows at time t hours. If we assume that the student starts (at time $t = 0$) knowing no skills, then $S(0) = 0$. But

$$S(0) = 10 \ln\big((0) + 1\big) + C = 10 \ln(1) + C = 10 \cdot 0 + C = C$$

  Therefore, $C = 0$.

So the simplified formula for $S(t)$ is the following function. (No longer just a function form.)

$$S(t) = 10 \ln(t+1)$$

This is the number of skills that the student knows at time t hours.

With that better understanding of what $S(t)$ means, we can now interpret the result of part **(a)**.

In part **(a)**, we found the unsigned area (USA) using the definite integral

$$USA = \int_1^3 S'(t)dt$$

But we realize that the same definite integral computes the signed area

$$SA = \int_1^3 S'(t)dt$$

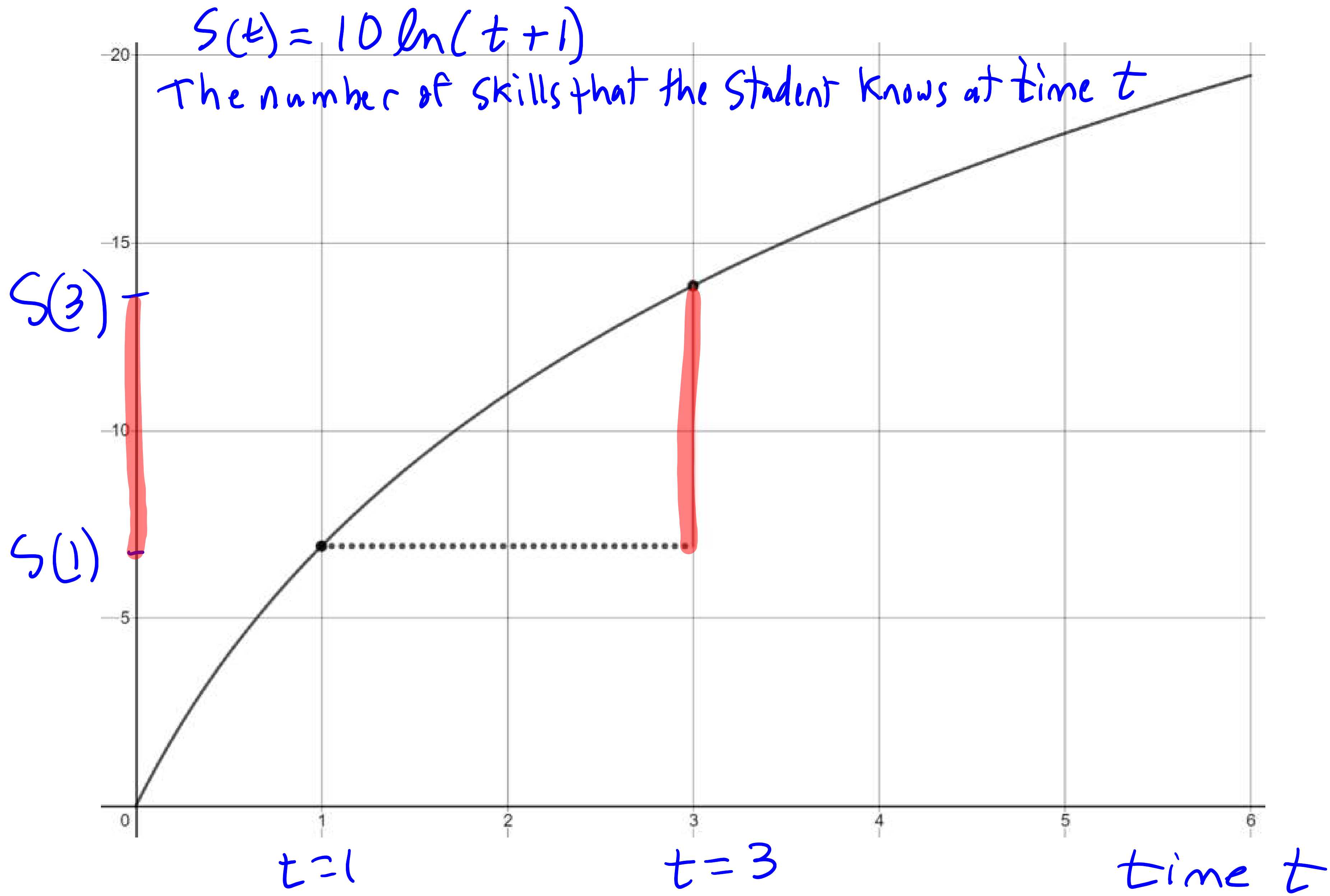
By the Fundamental Theorem of Calculus, the value of this integral equals the change in the number of skills known from $t = 1$ hours to $t = 3$ hours.

$$\int_1^3 S'(x) \underset{FTC}{=} S(3) - S(1)$$

That is, the number of skills learned from time $t = 1$ hours to time $t = 3$ hours is roughly 7.

That sentence is the interpretation of the result of part **(a)**.

That number can be illustrated on the graph of $S(t) = 10\ln(t + 1)$ shown on the next page.



End of [Example 2]