Subject for this video:

Total Income for a Continuous Income Stream

Reading:

- **General:** Section 6.2 Applications in Business and Economics
- **More Specifically:** bottom of page 400 – top of page 402, Example 3

Homework: H86: Total Income for a Continuous Income Stream (6.2#37,39*,41,43*))

Recall a Useful Indefinite Integral Rule

Exponential Function Rule #2 for Derivatives:	$\frac{d}{dx}e^{(kx)} = k \cdot e^{(kx)}$
Exponential Function Rule #2 for Indefinite Integrals:	$\int e^{(kx)}dx = \frac{e^{(kx)}}{k} + C$

And Recall the idea of Total Change Problems:

The definite integral can be used to find the total change of a function $F(x)$ from $x = a$ to $x = b$, if the derivative $F'(x)$ is known.

Definition of *Total Change Problems*

The Problem:

- **Given:** the rate of change of some quantity, $F'(x)$, and two numbers a, b with $a \leq b$,
- **Find:** the change in the quantity, $\Delta F = F(b) - F(a)$

Solution to the Problem: Use the Fundamental Theorem of Calculus

$$\Delta F = F(b) - F(a) \underset{FTC}{=} \int_a^b F'(x)\,dx$$

(This is simply a re-packaging of the Fundamental Theorem of Calculus.)

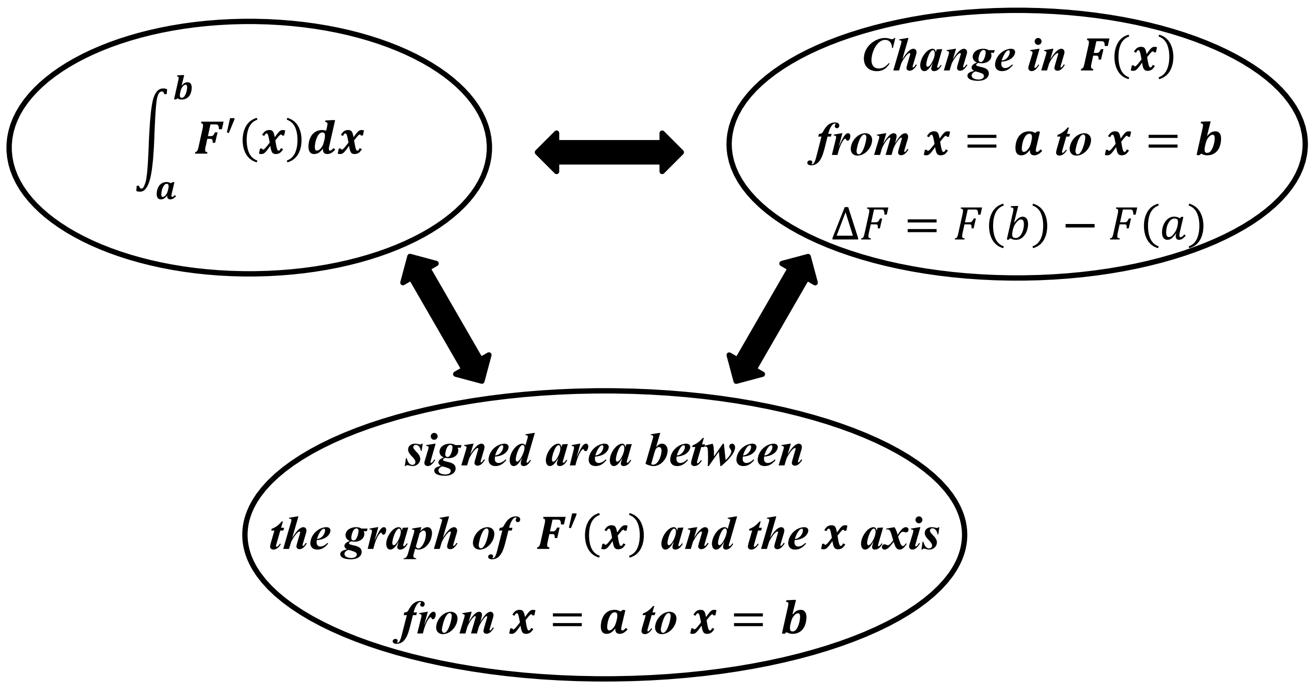
Finally, recall the discussion of Three Equal Quantities from the video for Homework H84:

Three Equal Quantities Related to *Area*

words: the ***change in $F(x)$ from $x = a$ to $x = b$***

words: the ***signed area between the graph of $F'(x)$ and the x axis from $x = a$ to $x = b$***

words and symbol: the ***definite integral of $F'(x)$ from a to b,*** $\displaystyle\int_a^b F'(x)\,dx$



Three Equal Quantities Related to Area

(An important concept in the fourth month of the course)

Continuous Income Stream

A particular kind of *Total Change* problem involves *income* from a *continuous income stream*. The term ***continuous income stream*** is an idealized way to think of a source of income. For example, in a job paying $120k per year (after taxes) in monthly paychecks, the income delivery would not be continuous, but rather would consist of twelve payments of $10k each. But for the simplest kinds of mathematical analysis, we would pretend that the income was arriving in a continuous stream at the rate of $120k/year.

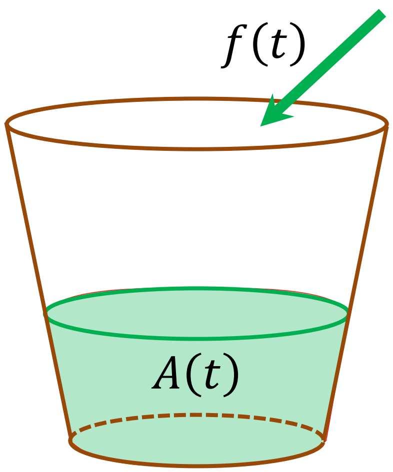
For many questions about income, the answers obtained by considering the income stream as continuous are exactly the same as the answers one would obtain by considering the income as arriving in discrete payments, and the answers are more easily obtained. And for questions where the answers are not exactly the same, the answers will still usually be close and, again, much more easily obtained. For example, when considering questions about an income stream that involve time scales that are long compared to the intervals of the discrete payments, it doesn't make much difference if one just treats the income stream as continuous.

In our examples, the ***flow rate*** of a continuous income stream will be denoted $f(t)$. In this expression, t is the time in years, and $f(t)$ is the flow rate, in units of dollars per year, at time t.

Total Income from a Continuous Income Stream

The simplest question one can ask about a continuous income stream is,

Question: What is the total amount of income that flows in during a time interval $\boldsymbol{a \leq t \leq b}$?



To answer this question, we will imagine that the income is accumulating in an account, with accumulated amount (in dollars) at time t denoted $A(t)$.

We can visualize this with a picture showing money flowing into a bucket.

With this notation, the total income TI during the time interval $a \leq t \leq b$ will just be the change in the accumulated amount $A(t)$. That is, the quantity $TI = \Delta A = A(b) - A(a)$.

In our picture, $TI = \Delta A$ will just be the change in the amount of money in the bucket.

The key to answering the question posed above is to note that the amount of money in the bucket is changing because money is flowing into the bucket. The *rate at which the amount of money in the bucket is changing* will be equal to the *rate of the flow*. That is,

$$A'(t) = f(t)$$

Therefore, we can find the answer to our question by integrating. Here is the process

$$TI = \Delta A = \underbrace{A(b) - A(a)}_{\text{total change in } A(t)} \underset{FTC}{=} \underbrace{\int_a^b A'(t)dt}_{\text{definite integral of } A'(t)} = \underbrace{\int_a^b f(t)dt}_{\text{because } A'(t)=f(t)}$$

= area between $f(t)$ and the $\cancel{x}$ t axis from ~~$t = 0$ to $t = T$~~ $t=a$ to $t=b$

It is worthwhile to summarize our findings in a nice green box.

Total Income for a Continuous Income Stream

If an Income Stream has a flow rate $f(t)$ that is a continous function on a time interval $[a, b]$, then the ***Total Income*** during the time interval $[a, b]$ is given by the definite integral

$$total\ income = TI = \Delta A = A(b) - A(a) = \int_a^b f(t)dt$$

This number can be visualized as the area of the region between the graph of the flow rate $f(t)$ and the t axis on the time interval $[a, b]$.

[Example 1] (Income Stream with Constant Flow)

(a) Find the total income produced by a continuous income stream in the first 10 years if the flow rate is

$$f(t) = 3000 \text{ } dollars \text{ } per \text{ } year$$ (constant flow)

Solution: We integrate the flow rate over the interval [0,10].

$$TI = \Delta A = \int_0^{10} f(t)dt$$

$$= \int_0^{10} 3000dt$$

$$\underset{FTC}{=} \left(\int 3000dt\right)\Big|_0^{10}$$

$$= (3000t + C)|_0^{10}$$

$$= \left((3000(10) + \cancel{C}) - (3000(0) + \cancel{C})\right)$$

$$= \left((30{,}000 + C) - (C)\right)$$

$$= 30{,}000$$

Indefinite Integral Details:

Rewrite the integrand first

$$f(t) = 3000 = 3000 \cdot 1$$

Then integrate

$$A(t) = \int 3000 \cdot 1dt$$

$$= 3000 \int 1dt$$

$$= 3000t + C$$

(b) Illustrate using a graph of $f(t)$ and a graph of $A(t)$.

Solution:

$A(t)$

It is worthwhile to start by considering what we know about the accumulated amount, in order to simplify the result of the indefinite integral. The result of the indefinite integral is the function form

$$A(t) = 3000t + C$$

which is the general antiderivative of $\cancel{I'(t)} = f(t) = 3000$

$A'(t)$

Observe that at time $t = 0$,

$$A(0)\ \cancel{I(0)} = 3000(0) + C = C$$

Therefore, the constant C represents the accumulated amount at time $t = 0$. But the problem statement says

"…*first* 10 *years*…"

so we can assume that the accumulated amount was 0 when the income stream started flowing at time $t = 0$. Therefore, the constant C must have the value $C = 0$, so that the accumulated amount is the function (*not* a *function form*)

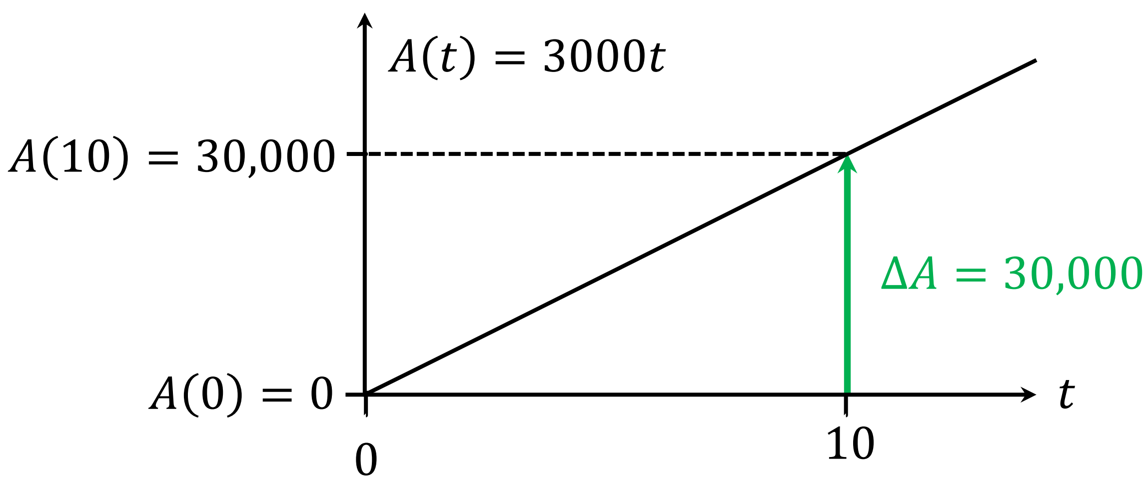
$$A(t) = 3000t$$

This function has the property that $A(0) = 0$.

The graph of $A(t)$ will be a line with slope $m = 3000$ and y intercept at $\big(t, A(t)\big) = (0,0)$. On this graph, the quantity

$$Total\ Income = TI = \Delta A = A(10) - A(0)$$

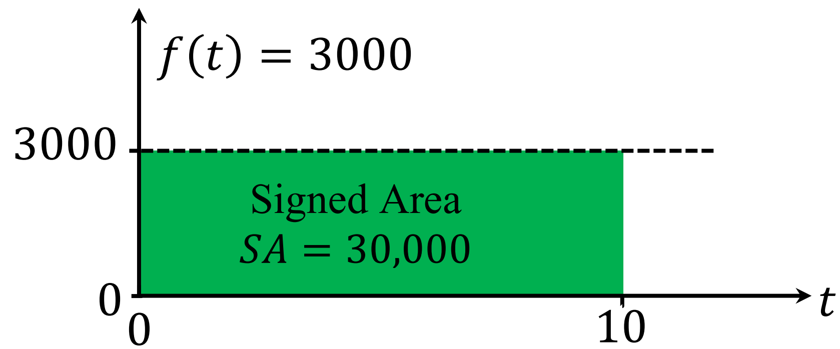
will appear as a change in *height*.



The graph of the flow rate $f(t) = 3000$ will be a horizontal line with line equation $y = 3000$. (Constant flow.) On this graph, the quantity

$$Total\ Income = TI = \Delta A = \int_0^{10} f(t)dt = \int_0^{10} 3000dt$$

will corresond to the *area* of a region between the graph of $f(t)$ and the t axis.



End of [Example 1]

Our next example of the total income from a continuous income stream has a flow rate that is *not* constant.

[Example 2] (Income Stream with Variable Flow)

(a) Find the total income produced by a continuous income stream in the first 10 years if the flow rate is

$$f(t) = 600e^{(0.06t)} \; dollars \; per \; year$$ (exponential flow)

Solution: We integrate the flow rate over the interval [0,10].

$$TI = \Delta A = \int_0^{10} f(t)dt$$

$$= \int_0^{10} 600e^{(0.06t)}dt$$

$$\underset{FTC}{=} \left(\int 600e^{(0.06t)}dt\right)\Big|_0^{10}$$

$$= \left(10{,}000e^{(0.06t)} + C\right)\Big|_0^{10}$$ (See Indefinite Integral Details on the Next Page)

$$= \left(10{,}000e^{(0.06(10))} + \cancel{C}\right) - \left(10{,}000e^{(0.06(0))} + \cancel{C}\right)$$

$$= 10{,}000e^{(0.6)} - 10{,}000e^{(0)}$$

$$= 10{,}000e^{(0.6)} - 10{,}000$$ exact

$$\approx 8221.18$$ approximate

Indefinite Integral Details:

$$A(t) = \int 600e^{(0.06t)}dt$$

$$= 600 \int e^{(0.06t)}dt$$

$$\int e^{(kt)}dt = \frac{e^{(kt)}}{k} + C$$

$$= \frac{600e^{(0.06t)}}{0.06} + C$$

$$= 10{,}000e^{(0.06t)} + C$$

(b) Illustrate using a graph of $f(t)$ and a graph of $A(t)$.

Solution: As we did in **[Example 1]**, it is worthwhile to start by considering what we know about the accumulated amount in order to simplify the result of the indefinite integral.

The result of the indefinite integral is the function form

$$A(t) = 10{,}000e^{(0.06t)} + C$$

which is the general antiderivative of $A'(t) = f(t) = 600e^{(0.06t)}$

Observe that at time $t = 0$,

$$A(0) = 10{,}000e^{(0.06(0))} + C = 10{,}000e^{(0)} + C = 10{,}000 + C$$

The amount $10{,}000 + C$ represents the accumulated amount at time $t = 0$. But the problem statement says "…*first* 10 *years*…" so we can assume that the accumulated amount was zero when the income stream started flowing at $t = 0$.

Therefore, the constant C must have the value $C = -10{,}000$, so that the accumulated amount is the function (*not* a *function form*)

$$A(t) = 10{,}000e^{(0.06t)} - 10{,}000$$

This function has the property that $A(0) = 0$.

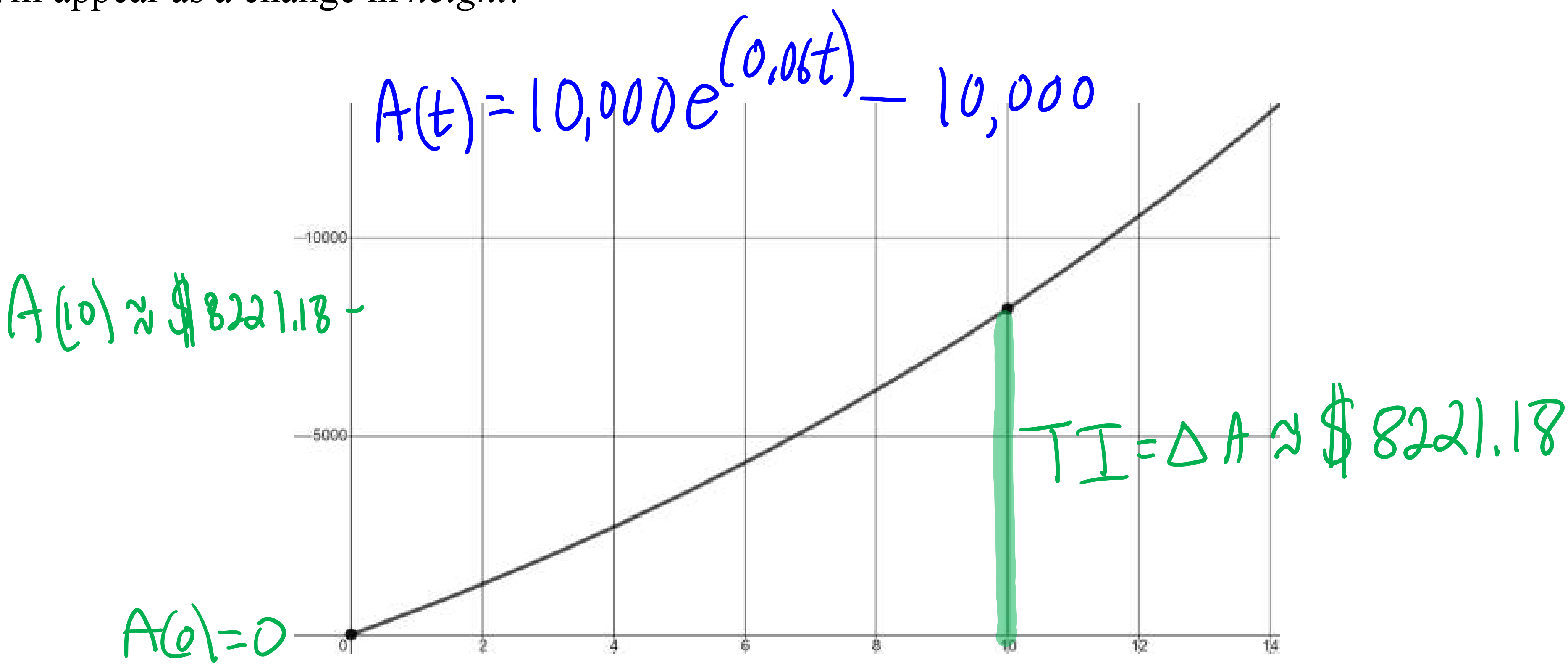
The graph of the accumulated amount

$$A(t) = 10{,}000e^{(0.06t)} - 10{,}000$$

will be an increasing exponential shape with y intercept at $(t, A(t)) = (0{,}0)$. On this graph, the quantity

$$Total\ Income = TI = \Delta A = A(10) - A(0) \approx 8221.18$$

will appear as a change in *height*.



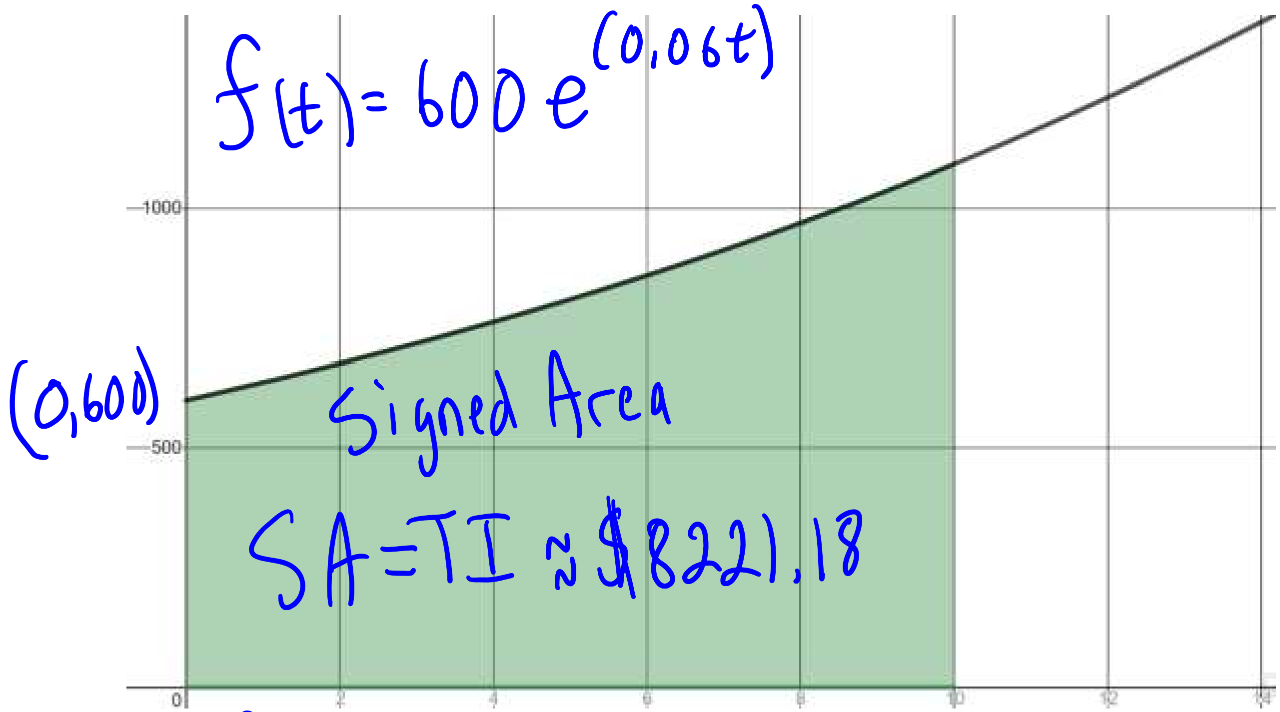
The graph of the flow rate

$$f(t) = 600e^{(0.06t)}$$

will be also be an increasing exponential shape, but with y intercept at $(t, f(t)) = (0{,}600)$. On this graph, the quantity

$$Total\ Income = TI = \Delta A = \int_0^{10} f(t)dt = \int_0^{10} 600e^{(0.06t)}dt$$

will corresond to the *area* of a region between the graph of $f(t)$ and the t axis.



End of [Example 2]