**Subject for this video:**

**Future Value and Present Value of a Continuous Income Stream**

**Reading:**

- **General:** Section 6.2 Applications in Business and Economics
- **More Specifically:** Top of page 402 – bottom of page 403 Example 4

**Homework:**

H87: Future Value and Present Value of a Continuous Income Stream (6.2#45,47,49,51,67)

**Recall the formula for Continuously Compounded Interest:**

**Continuously Compounded Interest Formula**

An account with ***continuously compounded interest*** has a balance described by the equation

$$A = Pe^{(rt)}$$

In this equation,

$P$ is the amount of the original deposit, called the *principal*.

$r$ is the *interest rate*, expressed as a decimal.

$t$ is the *time* in years since the original deposit.

$A$ is the *account balance* at time $t$.

## Recall a Useful Indefinite Integral Rule

| | |
|---|---|
| **Exponential Function Rule #2 for Derivatives:** | $\frac{d}{dx}e^{(kx)} = k \cdot e^{(kx)}$ |
| **Exponential Function Rule #2 for Indefinite Integrals:** | $\int e^{(kx)}dx = \frac{e^{(kx)}}{k} + C$ |

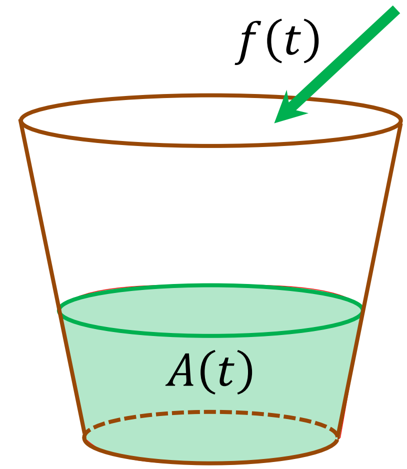
## Recall the idea of a Continuous Income Stream from the previous video

The term ***continuous income stream*** is an idealized way to think of a source of income. For example, in a job paying $120k per year (after taxes) in monthly paychecks, the income delivery would not be continuous, but rather would consist of twelve payments of $10k each. But for the simplest kinds of mathematical analysis, we would pretend that the income was arriving in a continuous stream at the rate of $120k/year. The ***flow rate*** of a continuous income stream will be denoted $f(t)$. In this expression, $t$ is the time in years, and $f(t)$ is the flow rate, in units of dollars per year, at time $t$.

**Also recall the idea of the Total Income from a Continuous Income Stream**

***Question: What is the total amount of income that flows in during a time interval $a \leq t \leq b$?***

To answer this question, we imagined that the income is accumulating in an account, with account balance (in dollars) denoted $A(t)$. We visualized this with a picture showing money flowing into a bucket. With this notation, the total income during the time interval $a \leq t \leq b$ will just be the change in the account balance $A(t)$. That is, the quantity $\Delta A = A(b) - A(a)$. In our picture, $\Delta A$ will just be the change in the amount of money in the bucket.




The answer to the question was found by integrating the flow rate.

**Total Income for a Continuous Income Stream**

If an Income Stream has a flow rate $f(t)$ that is a continous function on a time interval $[a, b]$, then the ***Total Income*** during the time interval $[a, b]$ is given by the definite integral

$$TI = \text{total income} = \Delta A = A(b) - A(a) \underset{FTC}{=} \int_a^b f(t)dt$$

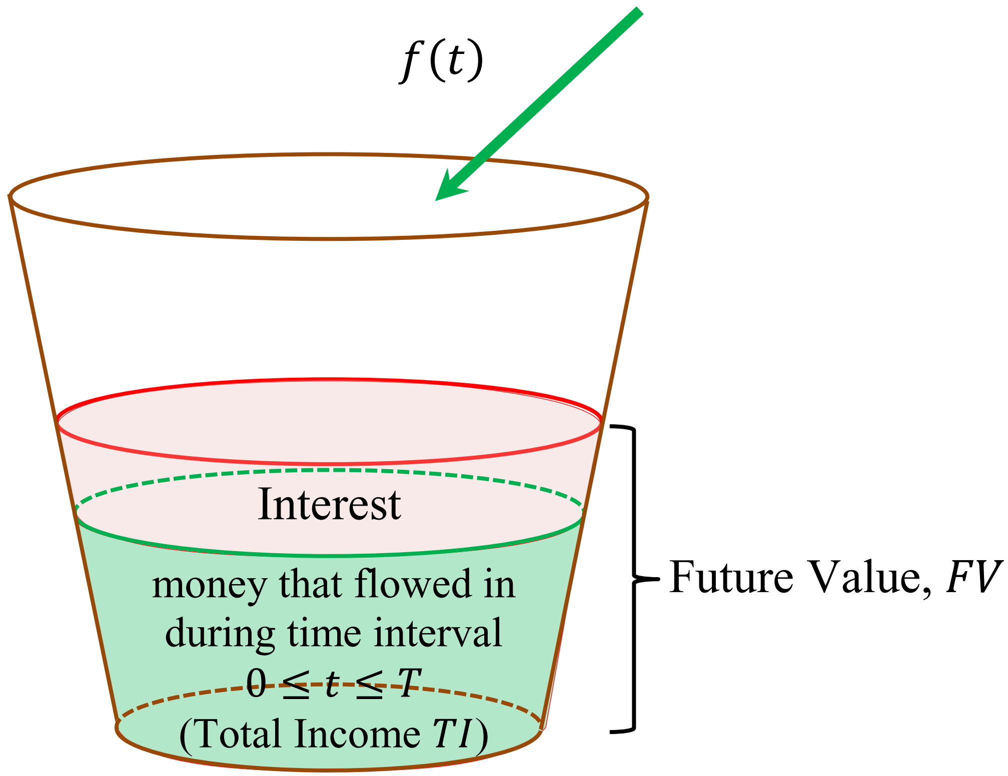
This number can be visualized as the area of the region between the graph of the flow rate $f(t)$ and the $t$ axis on the time interval $[a, b]$.

**Future Value of a Continuous Income Stream that Earns Interest**

In the discussion of Total Income from a Continuous Income Stream in the previous video, for the continuous income stream with flow rate $f(t)$, we were only concerned with the amount of money that accumulated in the bucket, the quantity $A(t)$.

Now consider a more complicated situation. Suppose that income stream with flow rate $f(t)$ flows into an account that earns continously compounded interest with interest rate $r$. As with the discussion of Total Income from a Continuous Income Stream, we can visualize this more complicated situation with a picture showing money flowing into a bucket.

Money that has flowed into the bucket (supplied by the continuous income stream) is shown in green. Once that money lands in the bucket, it starts earning interest, shown in red.




A natural question is,

**If a continuous income stream flows into an account that earns continuously compounded interest at a rate $r$, what will be the balance of the account at time $T$ years? (Note the capital $T$.) This balance is called the *Future Value at time T years*. (abbreviated *FV*)**

Finding an answer to this question is tricky. Note that the Continuosly Compounded Interest Formula

$$A = Pe^{(rt)}$$

is used in a situation where a single amount $P$ is deposited and a particular time $t$ elapses.

Consider the current situation, with money flowing into a bucket and then starting to earn interest only after it arrives in the bucket. Some of the money in the bucket will be in the bucket for a long time, earning interest for a long time. But some of the money will arrive in the bucket shortly before time $T$. That late-arriving money will not earn interest for very long.

In other words, there is not just a single deposit, and there is not just a single elapsed time.

It turns out that the answer to the question is a complicated-looking integral calculation:

**Future Value of a Continuous Income Stream**

If an Income Stream has a flow rate $f(t)$ that is continous on a time interval $0 \leq t \leq T$, and the income flows into an account that earns continuously compounded interest at rate $r$, then the *future value FV* at the end of $T$ years is given by the integral calculation

$$\text{future value} = FV = \int_0^T f(t)e^{(r(T-t))}dt$$

**Remark:** In the integrand, recognize the familiar term for the flow rate, $f(t)$. The other term in the integrand, $e^{(r(T-t))}$, is a term that accounts for the fact that different portions of the money in the bucket have been in the bucket for different amounts of time and as a result, will have earned interest for different amounts of time.

**[Example 1] Special Case: Future Value of a Continuous Income Stream with Constant Flow**

Suppose that a continuous income stream has constant flow rate $f(t) = c$ dollars per year and it flows into an account that earns continuously compounded interest at rate $r$. How much will be in the account at time $T$ years? That is, what is the Future Value $(FV)$ at time $T$ years?

**Solution**

The integral calculation follows on the next page

$$FV = \int_0^T f(t)e^{(r(T-t))}dt = \int_0^T ce^{(r(T-t))}dt \quad \text{(constant flow)}$$

$$\underset{FTC}{=} \left( \int ce^{(r(T-t))}dt \right) \Big|_0^T$$

$$= \left( -\frac{ce^{(r(T-t))}}{r} + E \right) \Bigg|_0^T \quad \text{(see details on next page)}$$

$$= \left( -\frac{ce^{(r(T-(T)))}}{r} + E \right) - \left( -\frac{ce^{(r(T-(0)))}}{r} + E \right)$$

$$= -\frac{ce^{(r \cdot 0)}}{r} + \frac{ce^{(r \cdot T)}}{r}$$

$$= -\frac{c \cdot 1}{r} + \frac{ce^{(r \cdot T)}}{r}$$

$$= \frac{ce^{(r \cdot T)}}{r} - \frac{c \cdot 1}{r}$$

$$= \frac{c\left(e^{(r \cdot T)} - 1\right)}{r}$$

Indefinite Integral Details

$$
\begin{aligned}
A(t) &= \int ce^{\left(r(T-t)\right)}dt \\
&= \int ce^{(rT-rt)}dt \\
&= \int ce^{(rT)}e^{(-rt)}dt \\
&= ce^{(rT)}\int e^{(-rt)}dt \\
&= ce^{(rT)}\left(\frac{e^{(-rt)}}{-r} + D\right) \\
&= ce^{(rT)} \cdot \frac{e^{(-rt)}}{-r} + ce^{(rT)} \cdot D \\
&= -\frac{ce^{(rT)}e^{(-rt)}}{r} + E \\
&= -\frac{ce^{(rT-rt)}}{r} + E \\
&= -\frac{ce^{\left(r(T-t)\right)}}{r} + E
\end{aligned}
$$

This result is useful enough that we should put it in a green box

**Future Value of a Continuous Income Stream with Constant Flow**

If a continuous income stream with constant flow rate $f(t) = c$ dollars per year flows into an account that earns continuously compounded interest at rate $r$, then the balance of the account at time $T$ years (that is, Future Value $(FV)$ at time $T$ years) will be

$$FV = \frac{c\left(e^{(r \cdot T)} - 1\right)}{r}$$

**[Example 2](similar to 6.2#45)**

Suppose you deposit $1500 a year into an account that earns 3.5% interest, compounded continuously. Treat the yearly deposits into the account as a continuous income stream.

**(a)** How much will be in the account 40 years later?

**Solution:**

We can use the result of **[Example 1]** with particular values

$$c = 1500 \text{ (constant flow rate)}$$

$$r = 0.035 \text{ (interest rate expressed as a decimal)}$$

$$T = 40$$

The result is

$$FV = \frac{c\left(e^{(r\cdot T)} - 1\right)}{r} = \underbrace{\frac{1500\left(e^{((0.035)\cdot 40)} - 1\right)}{0.035}}_{\text{exact}} \approx \underbrace{\$130{,}937}_{\text{approximate}}$$

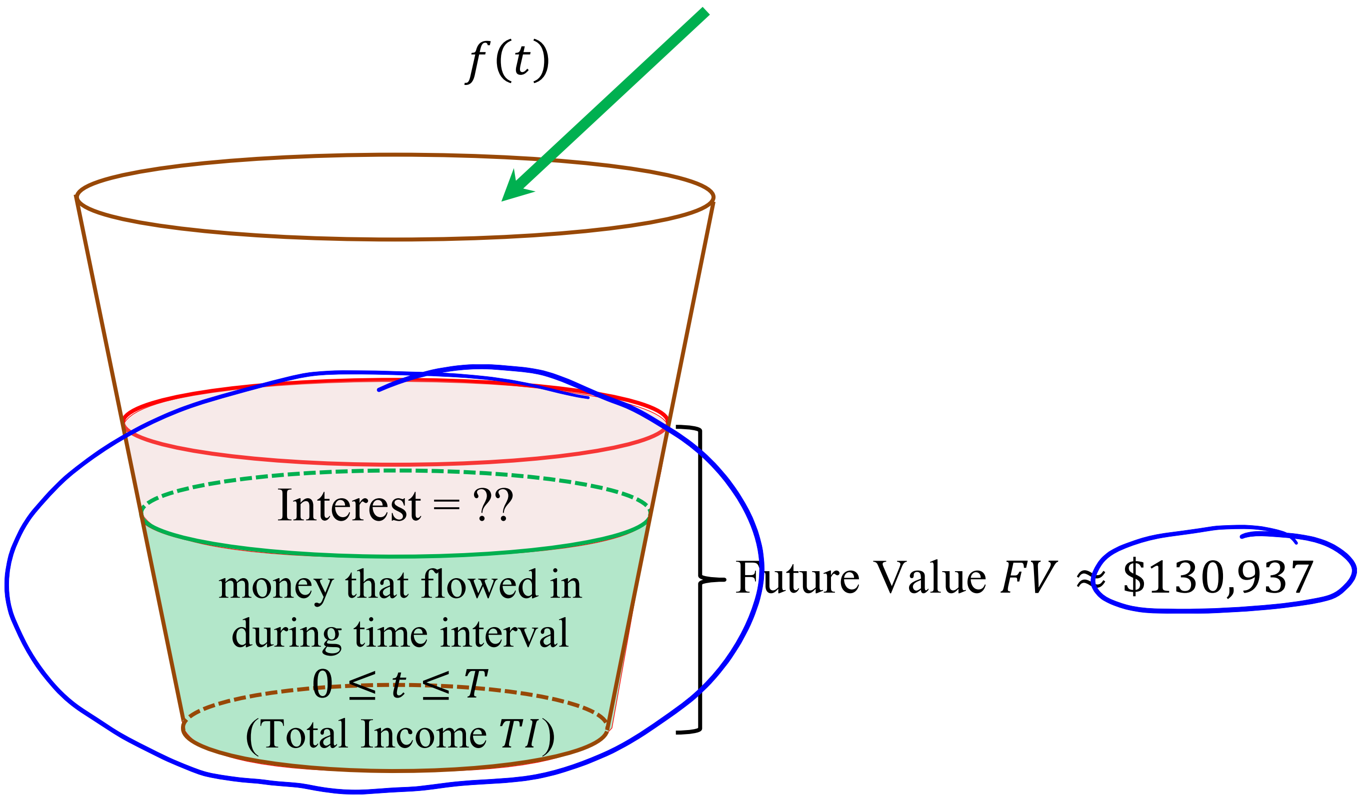
**(b)** How much of the final amount is interest?

**Solution:**

It will help to visualize this question using the bucket.

We have found that the Future Value is $FV \approx 130{,}937$.

We are being asked to find the portion of that amount that is interest.




Clearly, we simply need to subtract the Total Income $(TI)$ from the Future Value $(FV)$.

The total income is easy compute because the income stream has a constant flow

$$TI = \text{flow rate} \cdot \text{elapsed time} = c \cdot T = 1500 \cdot 40 = \$60{,}000$$

Therefore, the portion of the Future Value that is interest is

$$Interest = FV - TI \approx \$130{,}937 - \$60{,}000 = \$70{,}937$$

**[Example 3] Special Case: Future Value of a Continuous Income Stream with Exponential Flow**

Suppose that a continuous income stream has exponential flow rate

$$f(t) = c \cdot e^{(kt)}$$

dollars per year and it flows into an account that earns continuously compounded interest at rate $r$. How much will be in the account at time $T$ years? That is, what is the Future Value $(FV)$?

**Solution**

The integral calculation follows on the next page

$$FV = \int_0^T f(t)e^{(r(T-t))}dt = \int_0^T c \cdot e^{(kt)} \cdot e^{(r(T-t))}dt \quad \text{(exponential flow)}$$

$$= \left( \int c \cdot e^{(kt)} e^{(r(T-t))} dt \right) \Big|_0^T$$

$$\underset{FTC}{=} \left( \frac{ce^{(rT+(k-r)t)}}{k-r} + E \right) \Big|_0^T \quad \text{(see details on next page)}$$

$$= \left( \frac{ce^{(rT+(k-r)(T))}}{k-r} + E \right) - \left( \frac{ce^{(rT+(k-r)(0))}}{k-r} + E \right)$$

$$= \frac{ce^{(rT+kT-rT)}}{k-r} - \frac{ce^{(rT)}}{k-r}$$

$$= \frac{ce^{(kT)}}{k-r} - \frac{ce^{(rT)}}{k-r}$$

$$= \frac{c\left(e^{(kT)} - e^{(rT)}\right)}{k-r}$$

$$= \frac{c\left(e^{(rT)} - e^{(kT)}\right)}{r-k}$$

change order of subtraction
in both numerator
and denominator

Indefinite Integral Details

$$A(t) = \int c \cdot e^{(kt)} e^{\left(r(T-t)\right)} dt dt$$

$$e^{(rT-rt)} = e^{rT} e^{-rt}$$

$$= \int ce^{(kt)} e^{(rT-rt)} dt$$

$$= \int ce^{(rT)} e^{(kt-rt)} dt$$

$$= ce^{(rT)} \int e^{\left((k-r)t\right)} dt$$

$$= ce^{(rT)} \left( \frac{e^{\left((k-r)t\right)}}{k-r} + D \right)$$

$$= ce^{(rT)} \cdot \frac{e^{\left((k-r)t\right)}}{k-r} + ce^{(rT)} \cdot D$$

$$= \frac{ce^{(rT)} e^{\left((k-r)t\right)}}{k-r} + E$$

$$= \frac{ce^{(rT+(k-r)t)}}{k-r} + E$$

This was a very long, messy calculation. It is worthwhile to notice that the result actually agrees with the previous result for a special case. That is, if the exponential flow

$$f(t) = c \cdot e^{(kt)}$$

has $k = 0$, then the result simplifies to

$$FV = \frac{c\left(e^{(rT)} - e^{(kT)}\right)}{r - k} = \frac{c\left(e^{(rT)} - e^{((0)T)}\right)}{r - (0)} = \frac{c\left(e^{(rT)} - e^{(0)}\right)}{r} = \frac{c\left(e^{(rT)} - 1\right)}{r}$$

This is what we expect, because if $k = 0$, then the exponential flow is actually a constant flow, and the resulting expression for Future Value $(FV)$ agrees with the result from **[Example 1]**.

The result from the current example is useful enough that we should put it in a green box

**Future Value of a Continuous Income Stream with Exponential Flow**

If a continuous income stream with exponential flow rate $f(t) = c \cdot e^{(kt)}$ dollars per year flows into an account that earns continuously compounded interest at rate $r$, then the balance of the account at time $T$ years (that is, the Future Value $(FV)$ at time $T$ years) will be

$$FV = \frac{c\left(e^{(rT)} - e^{(kT)}\right)}{r - k}$$

**[Example 4] (similar to 6.2#47,49)**

**(a)** Find the future value, at 4.5% interest, compounded continuously for 10 years, of a continuous income stream with flow rate $f(t) = 1500e^{(-0.03t)}$ dollars per year. exponential flow

Give an exact answer and a decimal approximation, rounded to the nearest dollar.

**Solution:**

We can use the result of **[Example 3]** with particular values

$$c = 1500$$

$$k = -0.03$$

$$r = 0.045 \text{ (interest rate expressed as a decimal)}$$

$$T = 10$$

The result is

$$FV = \frac{c\left(e^{(rT)} - e^{(kT)}\right)}{r-k} = \frac{1500\left(e^{(0.045\cdot 10)} - e^{(-0.03\cdot 10)}\right)}{0.045-(-0.03)} = \frac{1500\left(e^{(0.45)} - e^{(-0.3)}\right)}{0.075} \approx 16{,}550$$

exact approximate

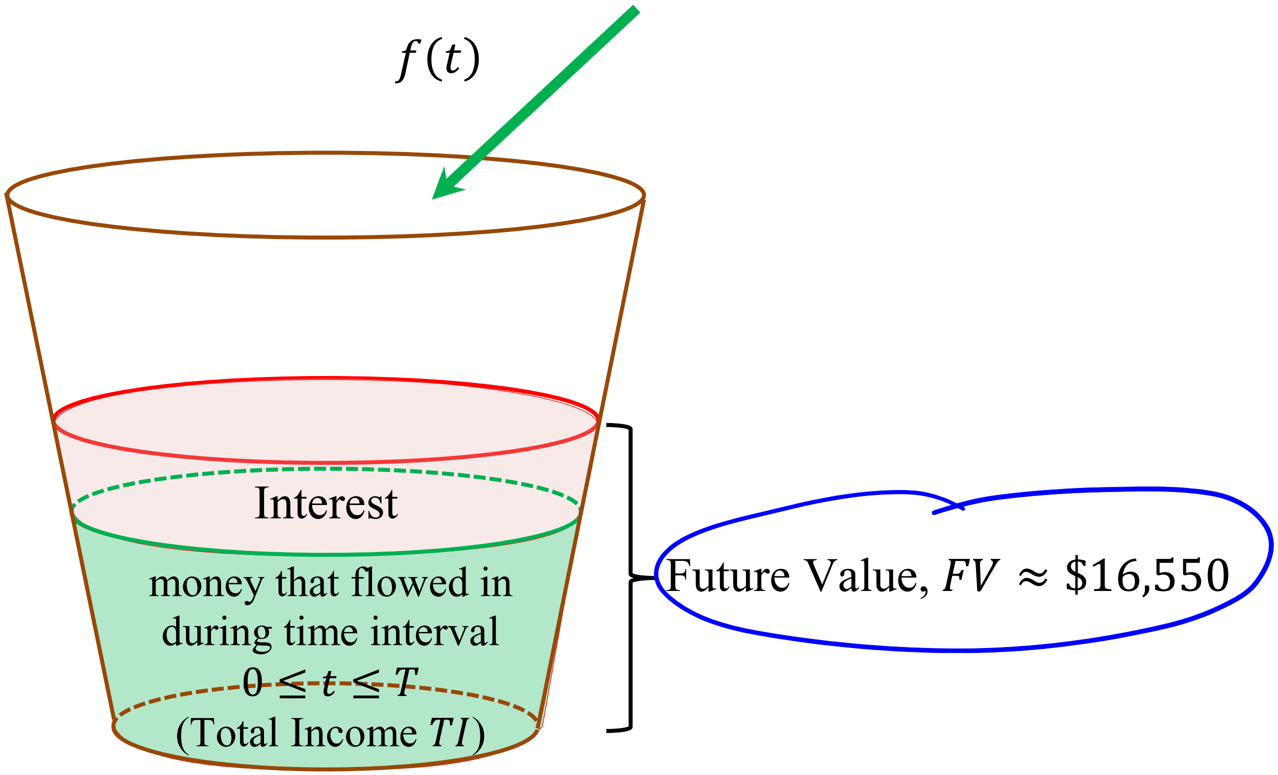
**(b)** Compute the portion of the future value in part **(a)** that is from money from the income stream and the portion that is from interest earned.

**Solution:**

As was the case in **[Example 2](b)** It will help to visualize this question using the bucket.

We have found that the Future Value is $FV \approx \$16{,}550$.

We are being asked to find the Total Income $(TI)$ and the Interest earned.



We start by computing the Total Income ($TI$) by integrating the flow rate over the interval [0,10].

$$TI = \int_0^{10} f(t)dt = \int_0^{10} 1500e^{(-0.03t)}dt$$

$$= \left(\int 1500e^{(-0.03t)}dt\right)\Big|_0^{10}$$

$$= \left(-50{,}000e^{(-0.03t)} + C\right)\Big|_0^{10} \quad \text{(see details on next page)}$$

$$= \left(-50{,}000e^{(-0.03(10))} + C\right) - \left(-50{,}000e^{(-0.03(0))} + C\right)$$

$$= -50{,}000e^{(-0.3)} + 50{,}000e^{(0)}$$

$$= -50{,}000e^{(-0.3)} + 50{,}000 \cdot 1$$

$$= 50{,}000 \cdot 1 - 50{,}000e^{(-0.3)}$$

$$= 50{,}000\left(1 - e^{(-0.3)}\right)$$

$$\approx \$12{,}959$$

**Indefinite Integral Details:**

$$A(t) = \int 1500e^{(-0.03t)}dt$$

$$= 1500 \int e^{(-0.03t)}dt$$

$$= \frac{1500e^{(-0.03t)}}{-0.03} + C$$

$$= -50{,}000e^{(-0.03t)} + C$$

We have found that the portion of the future value that is from the income stream is

$$TI \approx \$12{,}959$$

Therefore, the portion of the Future Value that is from Interest is

$$Interest = FV - TI \approx \$16{,}550 - \$12{,}959 = \$3591$$

**End of [Example 4]**

**[Example 5] (similar to Exercise 6.2#51)**

An investor is presented has a choice of two investments, Investment $A$ and Investment $B$, that can be bought for the same price.

Each investment produces a continuous income stream.

The flow rate of the income stream from Investment $A$ is $f(t) = 10{,}000$ dollars per year (constant flow)

The flow rate of the income stream from Investment $B$ is $g(t) = 9000e^{(0.03t)}$ dollars per year.
(exponential flow)

And both income streams will flow into an account earns 5% interest, compounded continuously.

Compare the future values of these investments to determine which is the better choice over the next 10 years.

**Solution:**

To compute the future values of the investments, we will use the formulas for

- the future value of a continuous income stream with constant flow (from **[Example 1]**)
- the future value of a continuous income stream with exponential flow (from **[Example 3]**)

Find the future value of Investment $A$, we use the result of **[Example 1]** with particular values

$$c = 10{,}000 \ \ \text{(constant flow rate)}$$

$$r = 0.05 \ \ \text{(interest rate expressed as a decimal)}$$

$$T = 10$$

The result is

$$FV = \frac{c\left(e^{(r\cdot T)} - 1\right)}{r} = \frac{10{,}000\left(e^{((0.05)\cdot 10)} - 1\right)}{0.05} = 200{,}000\left(e^{(0.5)} - 1\right) \approx \$129{,}744$$

exact    approximate

Find the future value of Investment *B*, we use the result of **[Example 3]** with particular values

$$c = 8000$$

$$k = 0.03$$

$$r = 0.05 \text{ (interest rate expressed as a decimal)}$$

$$T = 10$$

$f(t) = 8000e^{(0.03t)}$

The result is

$$FV = \frac{c\left(e^{(rT)} - e^{(kT)}\right)}{r-k} = \frac{8000\left(e^{(0.05\cdot 10)} - e^{(0.03\cdot 10)}\right)}{0.05 - 0.03}$$

$$= \frac{8000\left(e^{(0.5)} - e^{(0.3)}\right)}{0.02}$$

$$= 400{,}000\left(e^{(0.5)} - e^{(0.3)}\right) \quad \text{exact}$$

$$\approx 119{,}545 \quad \text{approximate}$$

Conclude that investment *A* is the better investment because its future value is greater.

## Present Value of a Continuous Income Stream that Earns Interest

When discussing the future value $(FV)$ of a continuous income stream that earns interest, we saw that an issue that complicated the computation of the future value was the fact that money that flows into the bucket ~~and~~ starts to earn interest with interest rate $r$ only after it arrives in the bucket. Some of the money in the bucket will be in the bucket for a long time, earning interest for a long time. But some of the money will arrive in the bucket shortly before time $T$. That late-arriving money will not earn interest for very long.

A natural question is,

> **How large would a one-time deposit need to be at time $\boldsymbol{t = 0}$, into an account earning continuously compounded interest with interest rate $\boldsymbol{r}$, to have the same value $\boldsymbol{(FV)}$ at the later time $\boldsymbol{T}$ as an account that has a continuous income stream, with flow rate $\boldsymbol{f(t)}$, also earning continuously compounded interest with interest rate $\boldsymbol{r}$ over the time interval $\boldsymbol{0 \le t \le T}$?**

The answer to the question is given a name, the *Present Value of the Continuous Income Stream*.

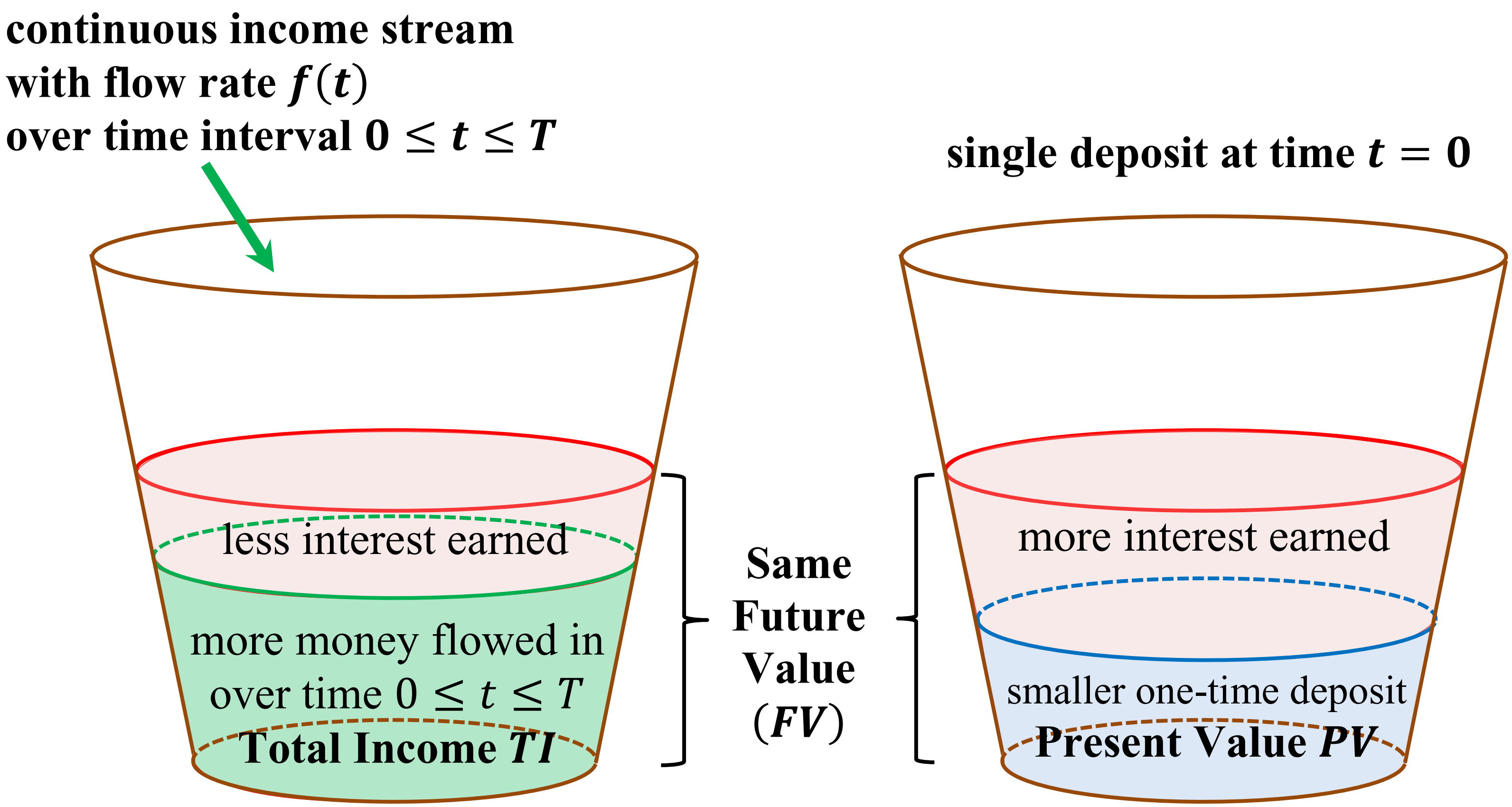
**Definition of the Present Value of a Continuous Income Stream**

**Words:** The Present Value $(PV)$ of a Continuous Income Stream

**Usage:** A continuous income stream with flow rate $f(t)$ flows for the time interval $0 \leq t \leq T$ into an account where it earns continuously compounded interest with interest rate $r$, resulting in future value $FV$ at time $t = T$.

**Meaning:** The Present Value $(PV)$ is defined to be the size of a one-time deposit made at time $t = 0$ into an account earning continuously compounded interest with interest rate $r$ that would result in an account balance equal to $FV$ at time $t = T$.

The diagram below illustrates the relationship between *Total Income*, *Future Value* $(FV)$, and *Present Value* $(PV)$ for a continuous income stream.




Observe that the Present Value $(PV)$, the size of the one-time deposit, does not need to be as large as the total income from the continuous income stream, because all of the money from the one-time deposit will sit in the bucket earning interest for the entire time $T$.

**[Example 6] (similar to 6.2#67)**

An investment will produce a continuous income stream with flow rate $f(t) = 12{,}000$ dollars per year for 10 years, flowing into an account where it will earn 6% interest compounded continuously.

**(a)** What is the Total Income of the income stream?

**Solution:**

The income stream has constant flow rate $f(t) = 12{,}000$ dollars per year.

If the income stream flows for 10 years, then the total income will be

$$\text{total income} = TI = 12{,}000 \cdot 10 = \$120{,}000$$

**(b)** What is the future value of the investment?

**Solution:**

The income stream has constant flow, so we can use the result of **[Example 1]** with particular values

$$c = 12{,}000 \;\; \text{(constant flow rate)}$$

$$r = 0.06 \;\; \text{(interest rate expressed as a decimal)}$$

$$T = 10$$

The result is

$$FV = \frac{c\left(e^{(r\cdot T)} - 1\right)}{r} = \frac{12{,}000\left(e^{((0.06)\cdot 10)} - 1\right)}{0.06}$$

$$= 200{,}000\left(e^{(0.6)} - 1\right) \quad \text{exact}$$

$$\approx \$164{,}424 \quad \text{approximation}$$

**(c)** What is the present value of the investment?

**Solution:**

The question is asking how big an initial deposit needs to be, into an account with 6% interest compounded continuously, for the account balance to be equal to

$$FV = \$164{,}424$$

after 10 years.

Recall that the Continuosly-Compounded Interest Formula is

$$A = Pe^{(rt)}$$

We know these values:

$$A = \$164{,}424$$

$$r = 0.06$$

$$t = 10$$

$$P = unknown$$

We need to find $P$, the amount of the initial deposit.

We start by solving the Continuously Compunded Interest Formula for $P$.

$$A = Pe^{(rt)}$$

$$P = \frac{A}{e^{(rt)}}$$

Then we substitute in our known values

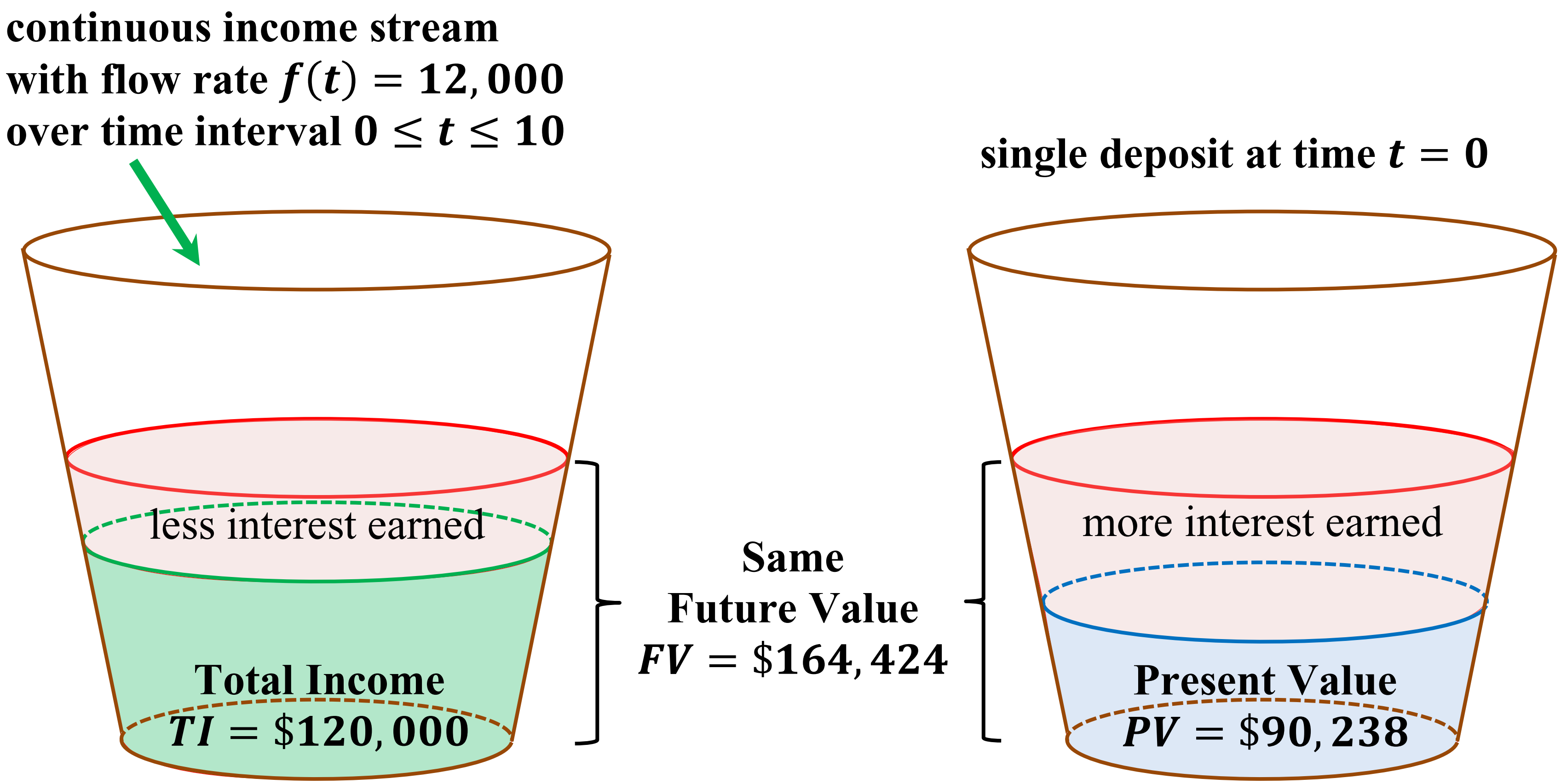
$$P = \frac{A}{e^{(rt)}} = \frac{164{,}424}{e^{(0.06 \cdot 10)}} = \frac{164{,}424}{e^{(0.6)}} \approx \$90{,}238$$

exact approximate

Therefore, the present value is

$$\text{present value} = PV \approx \$90{,}238$$

**Remark:** Notice that the present value is significantly smaller than the total income of the income stream




**End of [Example 6]**