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Can Infinite Repetitions Lead to Irrationality?

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It is common knowledge that repeating decimals are rational. What happens if we move to a different number system? This property certainly continues to hold in fixed-base systems such as binary, octal, and hexadecimal systems. However, in mixed-base systems where the base varies from digit to digit, we show that the exact opposite is true under a natural assumption on the bases. Central to our argument is a calculus-free technique from a recent elementary proof of the irrationality of e [**3**]. Our aims here are to highlight the contrasting impacts of repetitions on rationality in different number systems and to showcase how much mileage we can gain without resorting to explicit limit or series analysis.

Cantor expansions

In mixed-base systems, we generalize the choice of bases from a constant in fixed-base systems to a sequence of natural numbers. For the purpose of rationality analysis, we only need to expand the fraction part of a real number.

Definition. The *Cantor expansion* of a real number α in a given base sequence $\{b_n\}$ of natural numbers with $b_n \geq 2$ is $\alpha = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \ldots b_n}$, where $a_0, a_1, a_2, \ldots$ are integers with $0 \leq a_n < b_n$ for $n \geq 1$.

In a Cantor expansion, the sequence $\{a_n\}$ provides the digits in the mixed-base system with base sequence $\{b_n\}$. When $b_n = 10$ for all $n \geq 1$, the Cantor expansion reduces to the standard decimal expansion. When $b_n = n + 1$ for $n \geq 1$, the Cantor expansion generates the *factorial number system* [**8**]. In this system, $e = 2 + \sum_{n=2}^{\infty} \frac{1}{n!}$.

Every Cantor expansion is convergent, because the fraction part is bounded from above by the telescoping series $\sum_{n=1}^{\infty} \frac{b_n - 1}{b_1 b_2 \ldots b_n} = 1$; and every real number takes on a Cantor expansion representation [**10**]. Furthermore, if $\{b_n\}$ satisfies the condition that

$$\text{every prime divides infinitely many of the } b_n\text{'s,} \tag{1}$$

a Cantor expansion is irrational if and only if it is non-terminating, i.e., $a_n > 0$ and $a_n < b_n - 1$ hold infinitely often [**1**]. The base sequence in the factorial number system satisfies (1) clearly, and the irrationality of e follows. However, (1) is a sufficient but not necessary condition for irrationality [**4**], as we will see in an example later. In subsequent discussions we focus instead on base sequences that are *strictly increasing*, a characteristic shared by the factorial number system.

Repetitions lead to irrationality

We now present the following result about repeating Cantor expansions, which is proven by adopting the key technique in [**3**].

Theorem 1. *If $\{b_n\}$ is a sequence of strictly increasing natural numbers and $\{a_n\}$ is a repeating sequence of nonnegative integers which are not all zero, then the number $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \dots b_n}$ is irrational.*

Proof. Define a sequence $\{x_n\}$ by $x_n = \frac{a_n}{b_n} + \frac{a_{n+1}}{b_n b_{n+1}} + \frac{a_{n+2}}{b_n b_{n+1} b_{n+2}} + \cdots$. Since $\{a_n\}$ is a repeating sequence of nonnegative integers which are not all zero, all x_n's are positive. We observe that

$$\alpha = x_1 = \frac{1}{b_1}(a_1 + x_2) = \frac{1}{b_1}\left(a_1 + \frac{1}{b_2}(a_2 + x_3)\right) = \cdots . \tag{2}$$

If α is rational, (2) implies that all x_n's are also rational. Let $\frac{p_n}{q_n}$ be the irreducible fractional representation of x_n. Noting that

$$x_n = \frac{1}{b_n}(a_n + x_{n+1}), \tag{3}$$

we have $\frac{p_n}{q_n} = \frac{1}{b_n}\left(a_n + \frac{p_{n+1}}{q_{n+1}}\right)$, so $\frac{b_n p_n - a_n q_n}{q_n} = \frac{p_{n+1}}{q_{n+1}}$. Since the irreducible fraction $\frac{p_{n+1}}{q_{n+1}}$ has the smallest denominator possible, $q_n \geq q_{n+1}$, or

$$q_1 \geq q_2 \geq q_3 \geq \cdots . \tag{4}$$

Let's concentrate on the subsequence $\{x_{nt+1}\}_{n=0}^{\infty}$ where t is the period of $\{a_n\}$.

$$x_{nt+1} = \frac{a_{nt+1}}{b_{nt+1}} + \cdots + \frac{a_{nt+t}}{b_{nt+1} \dots b_{nt+t}} + \frac{a_{(n+1)t+1}}{b_{nt+1} \dots b_{nt+t} b_{(n+1)t+1}} + \cdots . \tag{5}$$

As n increases, the numerators in (5) remain the same repeating sequence of $a_1, \dots, a_t, a_1, \dots$ because of the repeating pattern of $\{a_n\}$, but the denominators in (5) all increase because of the monotonicity of $\{b_n\}$. Thus,

$$x_1 > x_{t+1} > x_{2t+1} > \cdots > 0. \tag{6}$$

In other words, $\frac{p_1}{q_1} > \frac{p_{t+1}}{q_{t+1}} > \frac{p_{2t+1}}{q_{2t+1}} > \cdots > 0$. Combining this with (4), we conclude that $p_1 > p_{t+1} > p_{2t+1} > \cdots > 0$, i.e., $\{p_{nt+1}\}_{n=0}^{\infty}$ is a strictly decreasing infinite sequence of positive integers. This is impossible. □

Remark. This theorem continues to hold even if the base sequence $\{b_n\}$ is increasing, but not strictly, as long as it does not stay constant after a certain point. Also the monotonicity of $\{b_n\}$ and the periodicity of $\{a_n\}$ only need to happen for all sufficiently large n. The proof above prevails with minimal modification.

As examples of Theorem 1, we re-establish the irrationality of the following numbers from [**4**].

1. $a_0 = 1$, $a_n = 1$ and $b_n = 2n(2n+1)$ for $n \geq 1$: $\sinh 1 = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$.
2. $a_0 = 1$, $a_n = 1$ and $b_n = 2n(2n-1)$ for $n \geq 1$: $\cosh 1 = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n)!}$.

The next two examples are generated from the Bessel-Clifford function [**6**] $C_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(k+n)!}$ with integer $k \geq 0$.

3. $a_0 = 1$, $a_n = 1$ and $b_n = n(k+n)$ for $n \geq 1$:
$$k! \cdot C_k(1) = 1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot (k+1)(k+2)\ldots(k+n)}.$$
4. $a_0 = 1$, $a_n = 1$ and $b_n = 2n(k+n)$ for $n \geq 1$:
$$k! \cdot C_k\left(\frac{1}{2}\right) = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n \cdot n! \cdot (k+1)(k+2)\ldots(k+n)}.$$
5. $a_0 = 0$, $a_n = 1$ and $b_n = p_n$, the nth prime, for $n \geq 1$: $\epsilon = \sum_{n=1}^{\infty} \frac{1}{p_1 p_2 \ldots p_n}$. Note that the base sequence here does not meet (1).

Remark. [**4**] uses a recursive formula with arguments tailor-made to each of these numbers. In particular, ϵ requires some extra effort in analysis.

Now we limit our discussion to Cantor expansions in the same increasing base sequence. If α and β are irrationals whose Cantor expansions repeat with periods s and t, respectively, so is $\alpha + \beta$ because its Cantor expansion repeats with a period at most st. Thus, the set of irrationals with repeating Cantor expansions is closed under addition. This property is certainly not shared by the set of irrationals in general.

Finally, we establish the irrationality of an alternating version of Cantor expansions known as infinite *Pierce expansions* [**12**], where the "digit" sequence $\{a_n\}$ repeats in segments of 1 and -1. Every real number from $(0, 1]$ can be represented as an infinite or finite ($\{a_n\}$ consists of finitely many alternating 1 and -1) Pierce expansion [**5**].

Theorem 2. *If $\{b_n\}$ is a sequence of strictly increasing natural numbers, then the number $\alpha = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{b_1 b_2 \ldots b_n}$ is irrational.*

Proof. Define a sequence $\{x_n\}$ by
$$x_n = \frac{1}{b_n} - \frac{1}{b_n b_{n+1}} + \frac{1}{b_n b_{n+1} b_{n+2}} - \frac{1}{b_n b_{n+1} b_{n+2} b_{n+3}} + \cdots .$$
Since x_n is absolutely convergent, we can subtract pairwise and rewrite it as
$$x_n = \frac{b_{n+1} - 1}{b_n b_{n+1}} + \frac{b_{n+3} - 1}{b_n b_{n+1} b_{n+2} b_{n+3}} + \frac{b_{n+5} - 1}{b_n \ldots b_{n+5}} + \cdots = \sum_{j=1,3,5,\ldots} \frac{b_{n+j} - 1}{b_n \ldots b_{n+j}}.$$
For each $j = 1, 3, 5, \ldots$, because $\{b_n\}$ is strictly increasing, $b_{n+j} - 1 \geq b_n$. We now have $\frac{b_{n+j} - 1}{b_n \ldots b_{n+j}} \Big/ \frac{b_{n+1+j} - 1}{b_{n+1} \ldots b_{n+1+j}} = \frac{b_{n+j} - 1}{b_n} \cdot \frac{b_{n+1+j}}{b_{n+1+j} - 1} > \frac{b_n}{b_n} \cdot 1 = 1$,

i.e., every term in x_n after pairwise subtraction is greater than its corresponding term in x_{n+1}. Thus, $x_n > x_{n+1}$, or, $x_1 > x_2 > x_3 > \cdots > 0$. This shows that $\{x_n\}$ shares the same strictly decreasing property as $\{x_{nt+1}\}$ in (6). In addition, $\alpha = x_1 = \frac{1}{b_1}(1 - x_2) = \frac{1}{b_1}\left(1 - \frac{1}{b_2}(1 - x_3)\right) = \cdots$ and $x_n = \frac{1}{b_n}(1 - x_{n+1})$, which are structurally identical to (2) and (3), respectively. If α is rational, we can apply the same argument as the proof of Theorem 1 to generate a punitive strictly decreasing infinite sequence of positive integers. □

Theorem 2 yields the irrationality of $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ trivially, as well as those of $\sin 1 = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!}$, $\cos 1 = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!}$, and the alternating versions of examples after Theorem 1.

Further reading

In 1869 Cantor wrote the first in-depth exposition of mixed-base systems in [**1**]. Subsequent studies on Cantor expansions include [**2, 11, 13**] and more recently [**7, 9, 14**]. These works contain results that are stronger, but more challenging to obtain, than what is presented here.

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Summary. Using a calculus-free technique, we prove that repeating “decimals” in mixed-base number systems with an increasing base sequence are irrational.

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