Unfinished Business

A) Product Spaces / States

Good reference: Wikipedia Hilbert Space

1) Backtrack: Symbol $\mathcal{H}_i$ or H_i denotes Hilbert space i ~~(defined in situation)~~ like $\mathbb{R}^n$, $\mathbb{C}^n$

"$\mathbb{R}$'s" n-D Real or Complex Number Spaces

ℓ^2 $\mathcal{L}^2$ L^2 spaces: $\mathcal{L}^2$

space of "square integrable" functions

Elster Google: Wikipedia "L2" Space

We can "combine" spaces:

2) Direct Sum "$\oplus$"

3-D space: $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}^1$

$\begin{pmatrix} a \\ b \end{pmatrix} \oplus (c)$

$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

(usually only makes sense if same "kind")

3) Direct or Tensor Product "$\otimes$"

6-D Space: $\mathbb{R}^3 \otimes \mathbb{R}^2 = \mathbb{R}^6$ "Side by side"

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} d \\ e \end{pmatrix}$

all combinations combine (a)(d), (a)(e) taken 2 at a time

If combine w/ regular multiplication

$a \otimes d = ad$

e.g. $3 \otimes 5 = 15$

$= \begin{pmatrix} a \otimes d \\ a \otimes e \\ b \otimes d \\ b \otimes e \\ c \otimes d \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} ad \\ ae \\ bd \\ be \\ cd \\ ce \end{pmatrix}$

QM Formalism of $H_1 \otimes H_2 = H_3$

Then form direct product (DP) ~~kets~~ kets

$$|\psi^{H_1 H_2}\rangle = |\psi^{H_1}\rangle|\psi^{H_2}\rangle$$
$$(= |\psi^{H_1}\rangle \otimes |\psi^{H_2}\rangle)$$

$$\mathrm{Dim}(H_3) = \mathrm{Dim}(H_1)\,\mathrm{Dim}(H_2)$$

Example. Next Slide

1) DP Basis + States

- Can always choose DP ket (Basis) ~~the basis~~ for H_3
- or linear combinations thereof
- Can NOT always write State $|\psi^{H_3}\rangle$ as DP Ket. $|\psi^{H_3}\rangle = |\psi^{H_2}\rangle|\psi^{H_1}\rangle$

Two Classes (States)

1) Seperable State: can write

2. Entangled State: can NOT

Number I in the World Example:

$$|\psi^{AB}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_A|-\rangle_B - |-\rangle_A|+\rangle_B\right)$$

Entanglement: more later.

Example 7

e.g. 2 spinor spaces (2 independent fermions)

$|S_1 S_2\rangle = |S_1\rangle |S_2\rangle$; 2 - 2D spaces

$\pm$ 2-D $\quad$ $\pm$ 2-D

$\mathrm{Dim}(|S_1 S_2\rangle) = 4$
$2 \otimes 2$

example orthonormal basis

- $|+\rangle|+\rangle \dots$
- $|+\rangle|-\rangle$

Note: order doesn't matter
better to think in 2 dim!
example: no associative "splitting" of direct product
(be careful)

$\langle \psi(p_1) | \quad |S_1^+\rangle |S_2^+\rangle$

e.g. $|+-\rangle$ & $|++\rangle$ orthogonal

$\langle ++|+-\rangle$ $\quad$ $(\langle +|\langle +|) \cdot (|+\rangle|-\rangle)$

$\langle +|$

MR of $|+-\rangle = |+\rangle|-\rangle$ (in ~~above~~ basis above)

$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ← location/order of components, arbitrary, just remain consistent

Example 2)
3D Pos/Mom Product

$|\vec{p}\rangle = |p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle$

$|x\rangle \overset{\text{def}}{=} |x_1\rangle |x_2\rangle |x_3\rangle$

B EPR : Paradox Einstein Podolsky Rosen

Sak. 2.3

Bells Theorem

1) Spin Correlation

Sequential Measurement of A, B ~~statistically~~ $\Rightarrow$ correlated.

- Correlation : $\langle ab \rangle \neq \langle a \rangle \langle b \rangle$

Experiment (loose): $C(x) = \frac{\langle ab \rangle (x)}{\langle a \rangle \langle b \rangle} \Rightarrow 1$

Ranges [1 to −1] 100% 100% anti

- Quantiki/Wiki $E(x) = \int P(x) X \, dx$

$\Rightarrow \sum \text{Prob}(x) X$

$= \langle x \rangle$

$S_z^A = S_z^A \otimes \mathbb{I}^B$ etc...

If Meas S_z^A :

If (+) collapse $|\psi^{AB}\rangle$ ~~...~~ $|+\rangle|-\rangle$

Meas S_z^B ? — w/ Corr = 100%

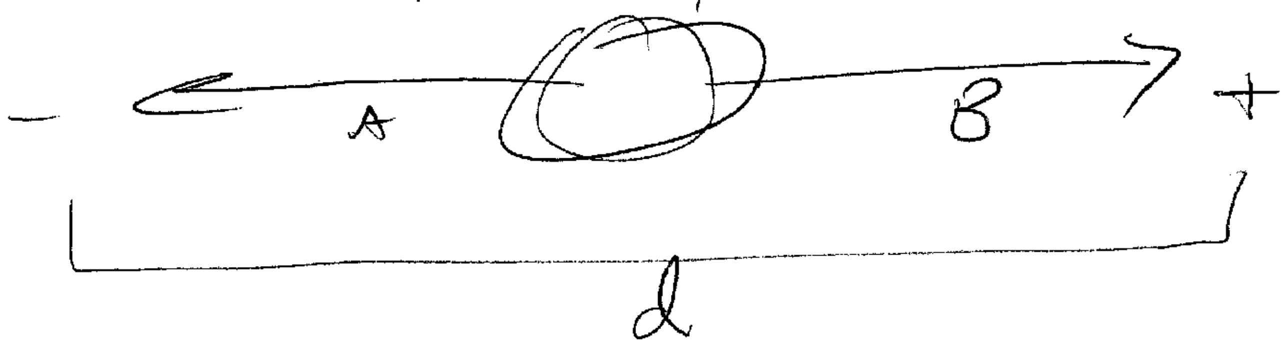
Similar if $S_z^A = ~~+~~$, $S_z^B = +$ ~~anti~~correl

(like sequential S_z w/ corr = 1)

2) Violation of Locality

Physics 612: together 2 fermions could represent Spin 0 system

2-Fermion Decay eg. $\eta = \mu^+ \mu^-$



Correlations independent of d!

As $d \to \infty$, Meas A affects Meas B at $v > c$

(Real Prob not "locality"
eg photons act at distance

If S^A_{8} +

Meas S^8_x ?

1+7 = 1/√2 (|8x7 + |5x7)

X+, X− random 50/50

Corr = 0

of Corr's

Make tables

See tables

A	B	Prob
Z+	Z−	1/2
Z−	X+/−	
X−	Z+/−	

A	B	Corr
a+	a−	1
a+	b+	Corr(ab) < 1
both	b−	

Bell's Inequality (ies)

Two ways to calc Probabilities (like those in table)

-1) "Hidden Variables" Theories

= ~~Se~~quence of Meas. predetermined: w/in sample i e.g. tables
- ~~Populations~~ $A(+)$ ~~Quant~~ sample Quantities ~~α~~ N_i

then e.g. Prob$(\hat{n}_A = \hat{a}_{+}, \hat{n}_B = \hat{c}_{+}) = \sum_{a+, c+} N_i \Big/ \sum_{all\, i} N_i$, $S_A = + \equiv A$

~~or~~ more generally $P_{AB}(\hat{n}_A, A, \hat{n}_B, B) = \dfrac{\sum_{n_A A,\, n_B B} N_i}{N}$

Easy to show b.c. $N_i \to$ positive def

$$\text{Prob}(a+, b+) \le \text{Prob}(a+, c+) + P(c+, b+)$$

- (QM → violated)

– More common approach: (web, Liboff p. 559)

e $\mathrm{Prob}(\hat{n}_A, A, \hat{n}_B, B) \Rightarrow \mathrm{Prob}(A(\hat{n}_A), B(\hat{n}_A)) \equiv \rho$

$\rho \Rightarrow \rho(\lambda)$ λ Hidden Variable

Define
Cs = $C(n_A, n_B) = \int A(\hat{n}_A) B(\hat{n}_B) \rho(\lambda)\, d\lambda$

e.g. more like "Corr" $(\sum \rightarrow \int)$

Bell's Inequality

$1 + C(n_A, n_B') \geq |C(n_A, n_B) - C(n_A, n_B')|$

Applies Generally for all cases n_A, n_B, n_B'

a bc   a, bc   abc

2) QM Calc of Probability

If $\hat{n}_A = \hat{a}$ + $\hat{n}_B = \hat{b}$, $A = +$ $B = +$ then it's same as meas $\hat{n}_A = \hat{a}+$ then meas $\hat{n}_A$ again but w $\hat{n}_A = \hat{b}-$

So Prob $= \left| \langle + S_A \cdot \hat{a} | S_A \cdot \hat{b} - \rangle \right|^2$

Easiest to solve w/ Pset input (e.g. Sak 1.23)

$|S \cdot \hat{n}+\rangle = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2} e^{i\alpha}|-\rangle$

$|S \cdot \hat{n}-\rangle = -\sin\frac{\beta}{2} e^{-i\alpha}|+\rangle + \cos\frac{\beta}{2}|-\rangle$

$\sqrt{\text{Prob}} = -\cos\frac{\beta_a}{2} \sin\frac{\beta_B}{2} e^{i(\alpha_a - \alpha_b)} + \cos\frac{\beta_B}{2} \sin\frac{\beta_A}{2} \overset{\text{trig rel.}}{=} \sin\left(\frac{\beta_B - \beta_A}{2}\right)$

can always orient $\hat{a}, \hat{b}, \hat{z}$ such that $\alpha_a - \alpha_b = 0$ + $\beta_B - \beta_A = \Delta\theta$

e.g. choose $\hat{z} = \hat{a}$, x-z plane $\rightarrow \hat{b}$

$*$ C.C. $= c$

$= \sin^2\left(\frac{\beta_B - \beta_A}{2}\right) = \sin^2\frac{\theta_{AB}}{2}$

Another way:

Rotation U prob 1.23 (Sak)

$U(\hat{n}) = \mathbb{1}\cos\frac{\phi}{2} - i\,\sigma \cdot n \sin\frac{\phi}{2}$

then Prob $= \left| \langle + | U(\hat{z} \times \hat{a}) U(\hat{z} \times \hat{b}) | - \rangle \right|^2$

~~[illegible]~~ will have terms like

$$\langle + | (\sigma \cdot a)(\sigma \cdot b) | - \rangle$$

for which we can use relation:

$$(\sigma \cdot a)(\sigma \cdot b) = \vec{a} \cdot \vec{b} + i\, \sigma \cdot (\vec{a} \times \vec{b})$$

useful relation derived from

$$\sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k$$

Actually since more common approach involves calc'ing

$$C(a,b) \longrightarrow \text{Expectation Value } \langle S \cdot \hat{a}\, S \cdot \hat{b} \rangle$$

In that case we start out w/ such an expression

~~IV~~ Unfinished Bus. (cont)

C) Mixed States ~~+~~ Density $\hat{\rho}$ Op.

Consider Examples so far:
for Single Atom Ag

$\boxed{SG}$ $|\psi\rangle$ Ag $\longrightarrow$ $\boxed{S_z}$ $|+\rangle$ $\longrightarrow$ $\boxed{S_x}$ $|S_x+\rangle$ $|S_x-\rangle$? inherent QM uncertainty $\Rightarrow [A, B] \neq 0$

or Degen: Prob: 1.23 $A = a\begin{pmatrix} 1 & \\ & -\mathbb{I}_{2\times 2}\end{pmatrix}$ $B = b\begin{pmatrix} 1 & \\ & \sigma_2 \end{pmatrix} \to \lambda^B_{2,3} = \pm b$

SG-like

Degen $-a$ space (2, 3)

$|\psi\rangle$ Ag $\longrightarrow$ $\boxed{A}$ $|-a\rangle \to \boxed{B}$ $|-b\rangle$ $|+b\rangle$?
$+a$

LHS Uncertain? Maybe No 1st Meas? Not completely uncertain!

Postulates so far $\longrightarrow$ $|\psi\rangle$ starts as definite vector

1st Meas: (both examples) Probs Predetermined! $|\psi\rangle = a|+\rangle + b|-\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$
Prob =

2nd example: same for 2nd Measurement

(Meas $-a$ $\longrightarrow$ $\text{Proj}_{W=2\oplus 3}|\psi\rangle$ = still def. vect: $\begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix}$ in $2 \oplus 3$

$\text{Prob}_{-b} = \left| \langle -b| \begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix} \right|^2$

How to INSERT LHS Uncertainty? what $|\psi\rangle$?

- No single $|\psi\rangle$ gives $S_{x,y,+z} = \pm$ 50/50!
(total uncertainty in $\hat{S}_x$, $\hat{S}_y$, + S_z same time) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

1) QM Solution ; Mixed States

Mix $|\alpha_i\rangle$'s w/ weights w_i's. $\Rightarrow$ Beam Polarizatio

$|\alpha_i$'s$\rangle$ in all directions $\rightarrow$ Unpolarized

$$|\alpha\rangle_{mix} = \sum_i w_i |\alpha_i\rangle \Rightarrow \int d\hat{n}\, w(\hat{n}) |\alpha(\hat{n})\rangle$$

could be infinite # of i's [

(fortunately easier than that)

$\sum w_i$ or $\int d\hat{n}\, w(\hat{n}) = 1$;

2) Consider just 2-term sum:

e.g. "$|\alpha\rangle_{mix} = \frac{1}{2}|+\rangle + \frac{1}{2}|-\rangle$"

Define Avg:

$$[A] = \sum w_i \langle A\rangle_i$$

$$\text{Prob}(\hat{A}_n = a_n) = \sum_i w_i |\langle a_n|\alpha\rangle_i|^2 = \sum_i w_i P^{rob}_i(a_n)$$

Characterize w/ matrix M_{ni}?

$$\text{Prob}(z\pm) = \frac{1}{2}(1) + \frac{1}{2}(0) = \frac{1}{2}$$

$$\text{Prob}(x\pm) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\pm\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$$\text{Prob}(|\alpha_{mix}\rangle) = \text{Prob}\left(|\alpha'_{mix}\rangle = \frac{1}{2}|x+\rangle + \frac{1}{2}|x-\rangle\right)$$

$|\alpha_{mix}\rangle$ + $|\alpha'_{mix}\rangle$ Physically Indistinguishable

~~# of~~ ~~ensemble~~ i in mix arbitrary

→ better to characterize w/ $\mathrm{Dim}(H) \times \mathrm{Dim}(H)$ matrix

(Insensitive to ambiguity)

b) Density Operator $\hat{\rho}_{mix\,i} = \sum_i w_i |\alpha_i\rangle\langle\alpha_i|$

correctly accounts for ambig.

our example

$$\rho_{mix} = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| = \frac{1}{2}|x+\rangle\langle x+| + \frac{1}{2}|x-\rangle\langle x-|$$

How to use

$$\boxed{\text{Density Matrix } \rho = MR(\hat{\rho})}$$

$[A]$ = S,K Gyp

$\mathrm{Prob}(a_n) = \langle a_n|\rho|a_n\rangle$

I) Mixed States (Review)

$|\psi\rangle \rightarrow \boxed{S_z}$

No single $|\psi\rangle$ can provide "total" uncertainty ($S_{\hat n} = \pm$ 50/50 for all $\hat n$)

Mixed States: Mix $|\alpha_i\rangle$'s w/ w_i

"$|\alpha\rangle_{mix} = \sum_i w_i |\alpha_i\rangle$" → ≠ Linear comb of ... Notation #1: Distinguishing ... we will not feature

→ Probably OK but we avoid (Literature employs)

→ Not very useful $\langle\alpha_{mix}|\alpha_{mix}\rangle$

$\sum_i w_i = 1$

$P(A=a_n)\ \bar{P}_{prob}(A=a_n) = \sum_i w_i |\langle a_n|\alpha_i\rangle|^2 = \sum_i P_i(a_n) w_i$

$[A] = \sum_i w_i \langle\alpha_i|A|\alpha_i\rangle$

Instead use operator: (or (ρ) = Matrix) $\hat\rho_{mix} = \sum_i w_i |\alpha\rangle\langle\alpha|$ Has nice property: that

$[A] = \sum w_i$

$= \text{Trace}(\rho A) = tr(\rho A) =$ Sak Derivation in class

$\text{Prob}(\langle a_n|a_n\rangle) = \langle a_n|\rho|a_n\rangle$

(nth diagonal element)

$\langle a|\ |a\rangle$ Trace $(\rho) = 1$ Trace $(O) = \sum_{s} \langle s|O|s\rangle$

– Has nice (weird) property

$$\rho_{unpol\,z} = \tfrac{1}{2}|+\rangle\langle+| + \tfrac{1}{2}|-\rangle\langle-|$$

$$= \tfrac{1}{2}|S\cdot n+\rangle\langle S\cdot n+| + \tfrac{1}{2}|S\cdot n-\rangle\langle S\cdot n-|$$

$$= \rho_{unpol\,\hat{n}} = \underline{\text{unpol}}$$

This happens precisely

bec. QM superposition ~~property~~ of $|\psi\rangle$'s

Note pure state:

$$(i \to 1 = w_i = 1)$$

$$\rho = 1\,|\alpha_i\rangle\langle\alpha_i|$$

obviously $\rho^2 = \rho$

$$tr(\rho A) \to tr(\rho\rho) = tr(\rho) = 1$$

$$tr(\rho^2) = 1 \quad \text{Pure ensemble}$$

Measure of Mix $= tr(\rho^2) < 1$ Mixed

for $H_{spin\,\frac{1}{2}}$

$\mathrm{Tr}(\rho A) = [A]$

$\mathrm{Tr}(\rho^2)$ measure of how mixed

$= \cancel{1}$ Pure
$= \frac{1}{2}$ Maximally mixed (in SU(2))

ρ replaces w_i's

Philosophy

- Usually ~~Mix~~ / ρ applies to large ensembles; (Say ~~Only~~ ensembles) large # identical systems
- For single system ? ; no way to distinguish
  Entanglement papers
- Implications

VI Time Dependence

A) Conceptual Intro: (Really Big Pic)

QM: Time is parameter not observable (like $\vec{x}$)
(Imagine s instead of t)

$\mathcal{H}_i \xleftrightarrow{\text{same}} \mathcal{H}_i$

Op's $B(t_1) \xleftrightarrow{\text{(new?)}} B(t_2)$

$|\alpha\rangle$ $|b_1\rangle, |b_2\rangle \xleftrightarrow{\text{change?}} |\alpha(t_2)\rangle, |b_1(t_2)\rangle, |b_2 t_2$

$U_{t_1 t_2}$

Main ideas: 1) Something changes

2) First order: not $\mathcal{H}_i$ (Hilbert sp.)

So what changes? Some combo: (bottom stuff) Ops/Basis/States

- ~~Exact~~ combo depends on "Picture"
  = several mathematically equiv. views (coord frame or vectors?) (both?) changing?

→ Implies a $U_{t_1 \to t_2}$ Trans. Op: → Rotations in $\mathcal{H}_i$

(update fig)

Uses of U's so far

- 1) Diagonalization: "Passive or Temporary" ⇒ No change
- 2) Space Translation: $U = T(a)$ Active Rotation/Mod $|x\rangle \to |x+a\rangle$
  Use: derive $\hat{p}$ (did not connect to motion)

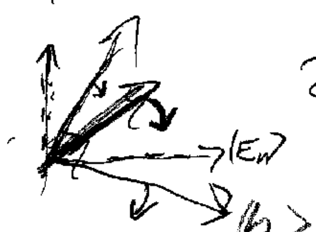
New 3) Time Translation: $U = \mathcal{U}(\Delta t) = \mathcal{U}_t$
- Active Rotations — System changes!!
- occurring continuously!

(Back to Picture)

What Properties for $\mathcal{U}_t$? First pick Depends on Picture

(Big Pic of Pictures)

- 1) Shrodinger ~~Picture~~ State Vector Rotate
  $|\alpha\, t_2\rangle = \mathcal{U}_{t_1 \to t_2} |\alpha\, t_1\rangle$
  (Active in all senses)



2) Heisenberg Op's/Basis Rotate
Passive? ought to be but No!
Because Absolute coordinate frame
$H \Rightarrow |E_n\rangle$ stays fixed
[Ham. determines $\mathcal{H}_i^{space}$]

Actually $|\alpha\rangle$ still moves w.r.t. $|E_n\rangle$ but $|b_n\rangle$ ~~const~~ move too ⇒ MAJOR

3) Other Pictures: Interaction Pics
- Both change (Ops-Basis vects)