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PRIME RINGS WITH A ONE-SIDED IDEAL SATISFYING A POLYNOMIAL IDENTITY

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It is known that the existence of a nonzero commutative one-sided ideal in a prime ring implies that the whole ring is commutative. Since rings satisfying a polynomial identity are natural generalizations of commutative rings the question arises as to what extent the above mentioned result can be extended to include these generalizations. That is, if R is a prime ring and I a nonzero one-sided ideal which satisfies a polynomial identity does R satisfy a polynomial identity?

This paper initiates an investigation of this problem. A counter example, given later, will show that the answer to the above question may be negative, even when R is a simple primitive ring with nonzero socle. The main theorem of this paper is Theorem 3 which states:

Let R be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that R satisfy a polynomial identity is that R have zero right singular ideal and $\hat{R}$, the right quotient ring of R, have at most finitely many orthogonal idempotents.

2. In the following given a ring R, $R^{\Delta}({}^{\Delta}R)$ denotes the right (left) *singular ideal* of R. Thus $R^{\Delta} = \{x \mid x \in R, x^r \in L^{\Delta}(R)\}$ where $L^{\Delta}(R)$ denotes the set of right ideals of R that meet, in a nonzero fashion, all right ideals of R. Similarly for ${}^{\Delta}R$ and ${}^{\Delta}L(R)$.

If Q is a ring such that R is a subring of Q and $qR \cap R \neq 0$ for each $q \in Q$ then Q is called a right quotient ring for R. Moreover if $Q = \{ab^{-1} \mid a, b \in R, b \text{ regular}\}$ then Q is called a classical right quotient ring. Following [2] we say that a ring R is right quotient simple if and only if it has a classical right quotient ring Q with $Q \cong D_n$, D_n a ring of $n \times n$ matrices over a division ring D.

From [4] we know that if R is a prime ring with $R^{\Delta} = 0$ then R has a unique maximal right quotient ring $\hat{R}$ where $\hat{R}$ is a prime regular ring. Moreover, letting $L(R)$ denote the lattice of right ideals of R, there is a mapping s: $A \to A^s$ of $L(R)$ which is a closure operation satisfying $0^s = 0$, $(A \cap B)^s = A^s \cap B^s$ and $(x^{-1} A)^s = x^{-1} A^s$. The set $L^s(R)$ of closed ideals of R can be made into a lattice in a natural way and it is shown in [4] that $L^s(R) \cong L^s(\hat{R})$ under the mapping $A \to A \cap R$, $A \in L^s(\hat{R})$. We shall have occasion to use the following realization of $\hat{R}$. Let $E = \cup_{A \in L^{\Delta}(R)} \mathrm{Hom}_R(A, R)$. On E

define the relation, $\alpha \equiv \beta$ if for some $A \in L^{\prime}(R)$, $A \subseteqq \text{Dom}\,\alpha \cap \text{Dom}\,\beta$ and $\alpha(x) = \beta(x)$ for each $x \in A$. It is shown in [5] that $\equiv$ is an equivalence relation and that $E/\equiv$ is a ring and in fact is $\hat{R}$.

The above remarks apply similarly to a prime ring R for which ${}^{\prime}R = 0$.

3. In this section occur the basic results of this paper. We will have occasion to use the result of Posner [8] stating that if R is a prime ring with polynomial identity then $\hat{R}$ is a classical two-sided quotient ring having the same multilinear identities as R. That part of Posners argument that shows if R has a polynomal identity then so does $\hat{R}$ is a very complicated argument and we take this opportunity to present a simple alternative argument.

LEMMA 1. *Let R be a prime ring with polynomial identity. Then $\hat{R}$ has a polynomial identity.*

Proof. From Posner [8] we know that R has left and right quotient conditions and hence R is right quotient simple, with $\hat{R} \cong D_n$. By a theorem of Faith and Utumi [2] R contains an integral domain K with right quotient ring $\hat{K} \cong \hat{D}$. Since K satisfies a polynomial identity we have by Amitsur [1] that $\hat{K}$ also has a polynomial identity. Thus D, and hence D_n, is finite dimensional over its center; thus D_n, so $\hat{R}$, has a standard identity.

LEMMA 2. *Let R be a prime ring with $R^{\prime} = 0$, let $A \in L^{\prime}(R)$ and let $\alpha \in \text{Hom}_R(R, R)$, R considered as a right R-module. If $\alpha(A) = 0$ then $\alpha = 0$.*

Proof. Let $x \in R$; then we have that $x^{-1}A \in L^{\prime}(R)$. If $r \in x^{-1}A$ then $xr \in A$ and thus $\alpha(xr) = 0$. Since α is a right R-endomorphism, $\alpha(xr) = \alpha(x)\cdot r$: It follows that $\alpha(x)\cdot x^{-1}A = 0$, hence $x^{-1}A \subseteqq \alpha(x)^r$. Thus $\alpha(x)^r \in L^{\prime}(R)$ and so $\alpha(x) \in R^{\prime}$. Hence $\alpha(x) = 0$.

The following lemma is trivial in the case R contains a central element. Without a central element the proof is more involved.

LEMMA 3. *Let R be a prime ring with a polynomial identity. Then $\text{Hom}_R(R, R)$ has a polynomial identity, if $R^{\prime} = 0$.*

Proof. From Lemma 1 we know that $\hat{R}$ has a polynomial identity. Consider $\hat{R}$ realized as $\bigcup_{A \in L(R)} \text{Hom}_R(A, R)/\equiv$. For $\alpha \in \text{Hom}_R(R, R)$ let $\bar{\alpha}$ denote the equivalence class in $\hat{R}$ determined by α. The mapping $\alpha \rightarrow \bar{\alpha}$ is a homomorphism of $\text{Hom}_R(R, R)$ into $\hat{R}$. If $\bar{\alpha} = \bar{\beta}$ then for some $A \in L^{\prime}(R)$ $\alpha(x) = \beta(x)$, $x \in A$. Thus $(\alpha - \beta)(A) = 0$. By Lemma 2 we see that $\alpha = \beta$. Thus $\alpha \rightarrow \bar{\alpha}$ is an injection onto a subring of $\hat{R}$ and so $\mathrm{Hom}_R(R, R)$ has a polynomial identity.

The following theorem provides a sufficient condition on the right ideal I having a polynomial identity to ensure the whole ring has a polynomial identity.

THEOREM 1. *Let R be a prime ring having a right ideal $I \neq 0$, I satisfying a polynomial identity and $I_l = 0$. Then R satisfies a polynomial identity.*

Proof. By assumption I_l, the left annihilator of I, is 0. Hence I is a prime ring itself. Considering I as a left I-module we have by the obvious dual of Lemma 3 that $\mathrm{Hom}_I(I, I)$, (the left I-endomorphisms), has a polynomial identity. For $x \in R$ the mapping $x \rightarrow r_x$, right multiplication by x, is an anti-isomorphism of R into $\mathrm{Hom}_I(I, I)$. Thus R itself satisfies a polynomial identity.

THEOREM 2. *Let R be a right quotient simple ring, $I \neq 0$ a right ideal of R satisfying a polynomial identity. Then R satisfies a polynomial identity.*

Proof. From Goldie [3] we have that I contains a uniform right ideal, thus we may assume I is uniform. Since $R^{\prime} = 0$ it follows that $\{x \mid x \in I,\ x^r \in L^{\prime}(R)\} = 0$, hence from [6] we have that $K = \mathrm{Hom}_R(I, I)$ is an integral domain. Moreover it is known ([3]) that $\hat{K} \cong D$, D a division ring, where $\hat{R} \cong D_n$. To complete the proof it suffices to show that D has a polynomial identity; the latter will hold provided K has a polynomial identity. To this end consider the homomorphism $a \rightarrow l_a$, left multiplication by a, of I into K. Let J denote the image of this map. $J = 0$ implies $I^2 = 0$ which is impossible; hence J is a nonzero subring of K satisfying a polynomial identity. Let $\alpha \in K$ and let $l_a \in J$. Let $x \in I$. Then $\alpha l_a(x) = \alpha(ax) = \alpha(a) \cdot x = l_{\alpha(a)}(x)$. Thus $\alpha l_a = l_{\alpha(a)} \in J$. Hence J is a left ideal of K. Since K is an integral domain we have by an obvious dual to Theorem 1 that K has a polynomial identity.

We now obtain, easily, the following.

THEOREM 3. *Let R be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that $\hat{R}$ satisfy a polynomial identity is that $R^{\prime} = 0$ and $\hat{R}$ have at most a finite number of orthogonal idempotents.*

Proof. Necessity is clear. Conversely, then, since $\hat{R}$ is regular with at most finitely many orthogonal idempotents it follows from [7] that $\hat{R}$ has the descending chain condition (d.c.c.) on right ideals. $\hat{R}$ is prime, thus $\hat{R} \cong D_n$ for some division ring D. Since $L^*(R) \cong L^*(\hat{R})$ we see that $L^*(R)$ has d.c.c. Thus from [4] we see that $\hat{R}$ is a classical right quotient ring, hence Theorem 2 applies.

The following example (communicated orally to S. K. Jain by A. S. Amitsur) shows that the extension of an identity from a right ideal to the entire ring is not always possible. Let F be a field and let F_∞ be the ring of all infinite matrices of finite rank. Let $a = (A_{ij})$ be a matrix such that $a_{11} \neq 0$ and $a_{ij} = 0$ for $i, j \neq 1$. Let $I = aF_\infty$. Then I satisfies the identity $(xy - yx)^2 = 0$ but F_∞ satisfies no identity at all.

4. Remarks.

In the case that R is primitive with a right ideal $I \neq 0$ having a polynomial identity then it is sufficient to assume that R has at most a finite number of orthogonal idempotents to ensure that R also have a polynomial identity.

There are other conditions one may impose upon R and I besides those given here, e.g. if R has at most finitely many orthogonal idempotents and I is a maximal right ideal or if $R^{\Delta} = 0$ and $I \in L^{\Delta}(R)$.

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