RINGS WITH CONSTRAINTS

By

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The problems on structure of rings with polynomial constraints, such as, polynomial identities, pivotal monomials, generalized polynomial identities, J-pivotal monomials, have recently found great interest with many authors. An algebra A over a field F is said to satisfy a polynomial identity if there exists an element $p(x_1, \ldots, x_t)$ of a free algebra over F generated by a set of non-commuting indeterminates $\{x_i\}$ such that $p(x_1, \ldots, x_t) = 0$ for all $x_i = a_i \in A$. The concept of pivotal monomials as given by Drazin [8] is as follows :

Let $\pi(x) = x_{i_1} \ldots x_{i_d}$ be a monomial in some non-commuting indeterminates $\{x_i\}$. Let $P_\pi = \{\sigma(x) = x_{j_1} \ldots x_{j_q} \mid$ either $q > d$ or if $q \leqslant d$ then $x_{j_h} \neq x_{i_h}$ for some $h \leqslant q\}$. Then a ring R is said to possess a right pivotal monomial $\pi(x)$ if for all $x = r \in R$, $\pi(r)$ is in the right ideal generated by $\{\sigma(r)_1\ \sigma(x) \in P_\pi\}$. It was shown by Drazin that if R satisfies a polynomial identity then R possesses a pivotal monomial.

The celebrated theorem of Kaplansky [13] for rings with polynomial identity (PI-rings) states that a primitive algebra satisfying a polynomial identity is finite dimensional over its centre.

The main structure theorem known for rings with pivotal monomials is that a primitive ring with a pivotal monomial is a full matrix ring over a division ring. Amitsur [4] has further generalized the concept of polynomial identities in other directions—J-pivotal monomials, generalized polynomial identities and so on. A ring R is said to have a J-pivotal monomial of degree d if its each primitive homomorphic image is a full matrix ring over a division ring of index $h \leqslant d$. A ring R which is an algebra over F is said to have a generalized polynomial identity if there exist an element $P(x)$ of the free product of the ring R and the free associative ring $F[x_1, x_2, \ldots \ldots]$ such that $P[x] = 0$ for every $x_i = r_i \in R$. It is shown by Amitsur [4] that a primitive ring R satisfies a generalized

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polynomial identity if and only if R is a dense ring of linear transformations of a vector space V_D over a division ring D such that the dimension of D over its centre C is bounded. The polynomial identities are the only polynomial constraints for which the structure theorems on prime rings satisfying these constraints have been given. Amitsur [1] has studied the structure of prime PI-rings by considering the polynomial ring $R[x]$ over a commutative indeterminate x and which has the same multi-linear polynomial identity that R has and is also semi-simple. Therefore the structure theorems for primitive rings enable us to obtain the structure of prime rings satisfying polynomial identities. In this regard we also mention here a beautiful recent result of Posner [16] which states that a prime PI-ring is a ring with Goldie conditions, *viz.*, (1) ACC hold on right as well as left annihilator ideals. (2) Any direct sum of right (or left) ideals contains only a finite no. of terms. Furthermore, Posner shows that the quotient ring which is a full matrix ring over a division ring also satisfies the same polynomial identity.

There has been so far no progress in the study of structure of prime rings with pivotal monomials, quasi-standard identities or quasi-pivotal-monomials. A ring R is said to satisfy a quasi standard identity of degree d if for each $x_1, \ldots\ldots\ldots, x_d \in R$ there exists an integer $n(x)$ such that $[S_d(x)]^{n(x)} = 0$ where $S_d(x) = \Sigma \pm x_1, \ldots\ldots x_d$ and the sign is +ve or −ve if the permutation is even or odd. One can similarly define quasi-pivotal monomial as generalization of pivotal monomials. The difficulty in this regard is that we cannot here reduce the problem to primitive rings whose structure is known to us as is done in the case of prime rings with polynomial identities. The reason why that approach fails is that if R has a pivotal monomial then $R[x]$ does not necessarily have a pivotal monomial and this follows from the fact that an infinite dimensional division algebra D has a pivotal monomial but $D[x]$ does not possess any pivotal monomial (cf. [2] and [11]). Professor Belluce of University of California and myself [7] have obtained a few results in this connection. For example we have proved : Let R be a prime ring with zero (right) singular ideal. Suppose R possesses uniform right ideals : (*i*) If R has a pivotai monomial of degree d then R is a right Goldie ring ; (*ii*) If R possesses a QSI of degree d then R is a PI-ring. We have also shown that an integral domain with a pivotal monomial is a right Ore-domain. This extends a similar result of Amitsur [3] known for PI-rings having no non-zero zero divisors.

The other type of problem on rings with constrains is that of localization. Let R be a ring. I be a non-zero one sided ideal with a constraint of some kind. Determine necessary and sufficient conditions on R and on I such that R satisfies some constraint.

Professor Belluce of University of California and myself [5] have shown that R satisfies a polynomial identity if and only if there is a non-zero one-sided ideal that is faithful (as a ring) and satisfies a polynomial identity. Another result is that a prime ring R satisfies a polynomial identity if and only if (0) there exists a non-zero right ideal I satisfying a polynomial identity (1) $R^{\triangle}$, the (rt) singular ideal, is zero and (2) $\hat{R}$, the maximal (rt) quotient ring of R, has at most a finite number of orthogonal idempotents. It has been shown by Belluce and myself [6] that if R is a primitive ring and $I \neq 0$, be a right ideal with a J-pivotal monomial then R has a non-zero socle and conversely.

The problem of determination of structure of rings in which a subset has some constraint is considered by Martindale [15] and recently by Kezlan [14]. Martindale proved that if A is a primitive algebraic algebra with involution whose symmetric elements satisfy a polynomial identity, then A is finite dimensional over its centre. Herstein conjecture that if A is a primitive algebra with involution whose symmetric elements satisfy a polynomial identity then S is finite dimensional over its centre is still open*. Kezlan has recently announced some results on rings having certain subsets which satisfy a polynomial identity cf [13]. No one has yet considered these questions when the subsets satisfy some polynomial constraint other than polynomial identity and it would be very interesting to obtain some structure theorems in this direction. For one such simple result on primitive rings a reference has already been made [6]. Even in the case of polynomial identities satisfied by certain subsets say one sided ideals, it is an interesting problem to obtain some other necessary and sufficient conditions (cf. [6]) on a ring and on an ideal such that the whole ring satisfies the same identity**.

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*Martindale has proved now that if R is an algebra with involution containing no non-zero nilpotent ideals whose sommetric elements satisfy a polynomial identity then R has PI. I take this opportunity to thank Professor Martindale for a preprint of his paper.

**For some more recent results in this connection refer the abstract in Notices Amer. Math. Soc. June 1967, p. 549, on Rings having one-sided ideals satisfying polynomial identities by S. K. Jain and Surjeet Singh.

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