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RINGS WITH FINITELY GENERATED INJECTIVE (QUASI-INJECTIVE) HULLS OF CYCLIC MODULES

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Let R be a ring. It is wellknown that if each cyclic R-module is injective, then R is semisimple artinian [6]. Rosenberg-Zelinsky [8] considered rings over which each cyclic R-module has injective hull of finite length. Osofsky [7] and Caldwell [2] studied hypercyclic rings and obtained their structure for semiperfect case. In this paper we have considered rings R over which every cyclic R-module has finitely generated injective hull, and we prove that if R is artinian, then each cyclic R-module has finitely generated injective hull iff each cyclic R-module has finitely generated quasi-injective hull iff the injective hull of the R-module R/J^2 is finitely generated (Theorem 2.4). In section 3, we show that if each cyclic R-module has cyclic injective hull or has cyclic quasi-injective hull, then R is artinian iff R has Krull dimension (Theorems 3.4, 3.7).

1. Notation and definitions. All rings considered have unity, and unless otherwise stated all modules are unital right modules. Let

R be a ring and let M be an R-module. $E_R(M)$ or (simply E(M)) will denote the injective hull of M. The quasi-injective hull of M will be denoted by q.inj.hull$_R$(M) (or simply q.inj.hull(M)).

For $X \subseteq R$, $\mathrm{ann}_M X$ will denote the right annihilator of X in an R-module M. If no confusion arises ann X will denote the right annihilator of X in R. J(R) (or J) will denote the Jacobson radical of R. $N \subset' M$ means that N is a large submodule of M. A ring R is called local if R has a unique maximal right ideal (which must be then J(R)). R is called right (left) valuation if the right (left) ideals of R are linearly ordered by set inclusion. R is called valuation if it is both right and left valuation. R is quasi-Frobenius iff R is right self-injective and right artinian. An R-module M is said to have finite uniform dimension if M contains only finite direct sums of nonzero submodules. M has finite uniform dimension n, denoted by dim M = n, if n is a positive integer such that $n = \sup\{\mathrm{card}(\Lambda: \oplus_{\alpha\in\Lambda} \Sigma M_\alpha$ is direct sum in M, $M_\alpha \neq (0)\}$. An R-module M is said to have Azumaya diagrams (AD) if $M = \oplus_{\alpha\in\Lambda} \Sigma M_\alpha$, where each R-submodule M_α has a local endomorphism ring. If $A \oplus B$ has AD and $A \oplus B \simeq C \oplus D$, then $A \simeq C$ implies $B \simeq D$. An ideal P is called completely prime if $xy \in P$ implies $x \in P$ or $y \in P$.

We now give the definition of Krull dimension in the sense of Rentchler-Gabriel, which was extended to infinite ordinals by Krause. Let M be a right R-module. The Krull dimension of M, denoted by K(M), is defined by transfinite recursion as follows: If M = 0, K(M) = -1; if α is an ordinal and $K(M) \nless \alpha$, then $K(M) = \alpha$ provided there is no infinite descending chain $M = M_0 \supset M_1 \supset \ldots$ of submodules M_i such that $K(M_{i-1}/M_i) \not< \alpha$, $i = 1,2,\ldots$. The Krull dimension K(R) of R is defined to be the Krull dimension of the right R-module R. We note that modules of Krull dimension 0 are precisely nonzero artinian modules, and that noetherian modules have Krull dimension. We also note that submodules and homomorphic images of a module with Krull dimension have Krull dimension. If R has Krull dimension, then finitely generated R-modules have Krull dimension. Further, $K(M) \leq K(R)$ whenever K(M) exists.

2. Finitely generated injective hulls.

We begin by stating the following known result (see [3], page 57; [1], page 286).

2.1. **LEMMA.** *Let R be any ring such that R/J is semisimple artinian and* $J \neq J^2$. *Then any simple R-module A that is not injective can be embedded in* J/J^2.

2.2. **LEMMA.** *Let M be an injective R-module and I,K be ideals in R, where* $K \subset I$. *Then* $\mathrm{Hom}_R(I/K,M) \simeq \frac{\mathrm{ann}_M K}{\mathrm{ann}_M I}$, *as R-modules.*

Proof: The proof follows from the canonical embedding of $\frac{\mathrm{ann}_M K}{\mathrm{ann}_M I}$ into $\mathrm{Hom}_R(I/K,M)$ and the Baer criterion for injective modules. □

2.3. **LEMMA.** *Let R be a ring such that every cyclic R-module has finitely generated q.inj.hull. Then every ring homomorphic image of R has this property.*

Proof: Let A be a two sided ideal of R. Let $\overline{R} = R/A$ and $\overline{R}/\overline{I}$ be a cyclic $\overline{R}$-module, where $\overline{I} = I/A$. $\overline{R}/\overline{I} \simeq R/I$ as R-modules. Denote by P the q.inj.hull of R/I as an R-module. P is a finitely generated R-module. Since $P = \mathrm{End}_R(E(R/I))R/I$, A annihilates P and hence P is an R/A-module and indeed P is quasi-injective as an R/A-module. If $B \subset P$, and B is quasi-injective as an R/A-module, then B is also quasi-injective as an R-module. Hence, P is the quasi-injective hull of R/I as an R/A module. Since P is a finitely generated R-module and A annihilates P, P is also finitely generated as an R/A-module. □

2.4 Theorem. *Let R be a right artinian ring. The following statements are equivalent.*

(a) Every cyclic R-module can be embedded in a finitely generated injective module.

(b) $\mathrm{Hom}_R(J/J^2, A)$ is finitely generated for every simple R-module A.

(c) Every cyclic R-module has finitely generated quasi-injective hull.

(d) The injective hull of R/J^2 is a finitely generated R-module.

Proof: (a) ⇒ (c): Let M be a cyclic R-module. Since q.inj.hull(M) $\subseteq$ E(M), and R is artinian, it follows that q.inj.hull(M) is also finitely generated.

(c) ⇒ (d): Let E_1 and E_2 denote respectively the injective hull of R/J^2 as R/J^2 and as R-modules. By Lemma 2.3, the quasi-injective hull of every cyclic R/J^2-module is finitely generated. In particular, q.inj.hull(R/J^2) (= E_1) is a finitely generated R/J^2-module. This implies $\mathrm{Hom}_{R/J^2}(J/J^2, I/J^2)$ is also finitely generated for any minimal right ideal I/J^2 of R/J^2 ([8], Theorem

1). But then $\mathrm{Hom}_K(J/J^2, I/J^2)$ is a finitely generated K-module, where $K = (R/J^2)/(J(R/J^2)) = (R/J^2)/(J/J^2) \simeq R/J$, (see Remark 1, [8]). Therefore, $\mathrm{Hom}_{R/J}(J/J^2, I/J^2)$ is a finitely generated R/J-module, and so $\mathrm{Hom}_R(J/J^2, I/J^2)$ is a finitely generated R-module for each simple submodule I/J^2 of R/J^2. This yields E_2, the injective hull of R/J^2 as an R-module, is finitely generated ([8], Theorem 1).

(d) $\Rightarrow$ (b): Let A be a simple R-module. If A is injective, then by Lemma 2.2 $\mathrm{Hom}_R(J/J^2, A) \simeq 0$, or A which is finitely generated. If A is not injective, then by Lemma 2.1, A can be embedded in J/J^2. So, there exists a right ideal I of R such that $I \subseteq J$ and $A \simeq I/J^2$. Since I/J^2 is a simple submodule of R/J^2, and the injective hull of R/J^2 as an R-module is finitely generated, $\mathrm{Hom}_R(J/J^2, I/J^2)$ is finitely generated. Therefore, $\mathrm{Hom}_R(J/J^2, A)$ is finitely generated. It remains to see that (b) $\Rightarrow$ (a), and this holds by Rosenberg-Zelinsky ([8], Theorem 1) and Lemma 2.1. □

3. Cyclic injective and quasi-injective hulls

In this section we consider rings with Krull dimension over which each cyclic module has cyclic injective hull (hypercyclic rings), or cyclic quasi-injective hull (q-hypercyclic rings). Indeed such rings turn out to be artinian rings. We shall need the following.

3.1. PROPOSITION. *If R is a ring with Krull dimension then $K(R) = K(R/P)$, for some prime ideal P. In fact P is a minimal prime.*

Proof: See ([4], Corollary 7.5). □

We give the proof of the following result.

3.2. LEMMA. *If R is a valuation ring, and $P = aR$ is a nonzero prime ideal with $P \neq J$, then P is not completely prime.*

Proof: Let $x \in J$, $x \notin P$. Then $Ra \subseteq Rx$. So, there exists $y \in R$ such that $a = yx$. Observe that $y \in J$; for if $y \notin J$, then $y^{-1}a = x$; that is, $x \in Ra$, which is a contradiction.

Suppose, on the contrary, that P is completely prime. Then $yx \in P$, $x \notin P$ implies $y \in P$. Thus there exists z such that $y = az$. Hence $a = azx$, $zx \in J$. That is, $a(1-zx) = 0$ and $1 - zx$ is invertible. Thus $a = 0$, which is a contradiction. Thus P is not completely prime. □

Before we give the main theorem in this section, we prove the following key result.

3.3. PROPOSITION. *Let R be a local hypercyclic ring. If R has Krull dimension, then R is right artinian.*

Proof: First suppose $J = J(R)$ is nil. Then, since R has Krull dimension, J is nilpotent. This implies J is the only prime ideal of R. But then by Lemma 3.1 $K(R) = K(R/J) = 0$, which proves that R is artinian.

Suppose J is not nil. Then by ([7], Theorem 2.12) there exists a nonzero nilpotent ideal $aR \subseteq J$, $a \in R$ such that aR is the maximal proper twosided ideal below J. Since R has Krull dimension, R satisfies acc on prime ideals. If J is not the only prime ideal, there exists a prime ideal Q such that Q is maximal among all the prime ideals different from J. Then $Q \subseteq aR$. Since aR is nilpotent, Q = aR. Thus Q ≠ (0), since aR is not zero. So by Lemma 3.2 and the fact that R is valuation [7], Q is not completely prime. Consider now the prime ring R/Q. Since R/Q has Krull dimension, it is a prime Goldie ring. Therefore Z(R/Q), the right singular ideal of R/Q, is zero. Thus ann $\bar{x} = 0$, for every $0 \neq \bar{x} \in R/Q$, since R is a valuation ring. Hence Q is completely prime, which is a contradiction. This proves J is the only prime ideal. Therefore, K(R) = K(R/J) = 0, and hence R is artinian. □

3.4. THEOREM. *Let R be a hypercyclic ring. Then R has Krull dimension iff R is artinian.*

Proof: If R has Krull dimension, then each homomorphic image of R has acc on direct summands. Thus, by Osofsky, R is a ring direct sum of matrix rings over local hypercyclic rings ([7], Lemma 1.7 and Theorem 1.18). Therefore, by Proposition 3.3, R is artinian. □

3.5. COROLLARY. *A hypercyclic ring with Krull dimension is quasi-Frobenius.*

Following the same method as for semi-perfect q-hypercyclic rings [5], we have the following:

3.6. SUBLEMMA. *Let R be a q-hypercyclic ring with finite uniform dimension. Then R is self-injective.*

This sublemma is used in the proof of the following theorem.

3.7. THEOREM. *Let R be a q-hypercyclic ring with Krull dimension. Then R is artinian.*

Proof: There exists a prime ideal P of R such that K(R/P) = K(R). Since R is q-hypercyclic, S = R/P is a q-hypercyclic ring ([5], Lemma 2.6). Therefore, S is right self-injective; that is, E(S) = S. Since S is a prime Goldie ring, Q(S) = E(S) = S. Thus S is artinian, that is K(S) = 0, which gives R is artinian. □

If R is a ring with Krull dimension such that the injective hull of every cyclic R-module is finitely generated or the quasi-injective hull of every cyclic R-module is finitely generated, we do not know whether R is artinian or not. But we have the following.

REMARK. *Let R be a ring with Krull dimension, and let P be a minimal prime ideal of R such that the prime ring R/P has a left classical quotient ring (In particular, if R has also Krull dimension as a left R-module). Then, if each cyclic R-module has finitely generated quasi-injective hull, R must be artinian.*

Proof of the Remark: Under our hypothesis, R/P is both right and left Goldie ring. Thus its right and left classical quotient rings coincide. Indeed the classical quotient ring is the injective hull of R/P as a right R/P-module. Since the injective hull $E_{R/P}(R/P)$ (= q.inj.hull_R(R/P)) is finitely generated, it follows that

$$Q(R/P) = a^{-1}b_1(R/P) + \dots + a^{-1}b_n(R/P), \text{ for some } a, b_1, \dots, b_n \in R/P.$$

Thus $aQ(R/P) = b_1(R/P) + \dots + b_n(R/P)$ so, Q(R/P) = R/P. Therefore, R/P is artinian. But then R is artinian. □

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