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# CONTINUOUS RINGS WITH ACC ON ESSENTIALS ARE ARTINIAN

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ABSTRACT. It is proved that a left continuous ring with ascending chain condition on essential left ideals is left artinian.

## 1. INTRODUCTION

It is well known that for left or right self-injective rings diverse chain conditions on certain classes of left ideals are actually equivalent to the minimum condition on the lattice of all left ideals. Recently, Huynh-Dung and Page-Yousif proved independently that a left or right self-injective ring with ascending chain condition on essential left ideals is left artinian. In this paper we prove the preceding result by assuming that $R$ is left continuous instead of being left self-injective. Examples are given to show that one-sided continuity and one-sided chain conditions may not necessarily yield the continuity or chain conditions on the opposite sides.

## 2. PRELIMINARIES

As defined by Utumi, a ring $R$ is called left continuous if (i) every left ideal of $R$ is essential in a direct summand of $R$ and (ii) every left ideal isomorphic to a direct summand of $R$ is itself a direct summand [6]. Continuous modules are defined analogously. Throughout, all rings have identity element and all modules are unital. $Z({}_RR)$, $J(R)$, and $\mathrm{Soc}({}_RR)$ will denote, respectively, the left singular ideal, the Jacobson radical, and the left socle of a ring $R$.

We record below some well-known results often referred to in the proof of our theorem.

**Lemma 1.** *If $R$ has acc on essential left ideals, then $Z({}_RR)$ is nilpotent.*

*Proof.* The proof is exactly the same as the one when $R$ has acc on left annihilators ([5, Lemma 3.39, p. 380]).

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**Lemma 2.** *Let $R$ be a left continuous ring. Then $J(R) = Z({}_RR)$, $R/J(R)$ is a von Neumann regular left continuous ring, and idempotents modulo $J(R)$ can be lifted.*

*Proof.* See Utumi [6].

The following lemma has recently been proved independently by Huynh-Dung and Page–Yousif. For completeness we include here its proof by Huynh-Dung [2].

**Lemma 3.** *For each left module $M$, $M$ has acc on essential submodules if and only if $M/\operatorname{Soc}(M)$ is left noetherian.*

*Proof.* If $M$ has acc on essential submodules, each submodule $N$ of $M$ has the same property. Therefore, if $B \subset' N \subset M$, $N/B$ is noetherian. In particular, it follows that every uniform submodule of $M$ is noetherian. Consider a complement $H$ of $\operatorname{Soc}(M)$. Then $\operatorname{Soc}(M) \oplus H \subset' M$, and therefore $\frac{M}{\operatorname{Soc}(M)\oplus H}$ is noetherian. Since

$$\frac{M}{\operatorname{Soc}(M)\oplus H} \simeq \frac{\frac{M}{\operatorname{Soc}(M)}}{\frac{\operatorname{Soc}(M)\oplus H}{\operatorname{Soc}(M)}} \quad \text{and} \quad \frac{\operatorname{Soc}(M)\oplus H}{\operatorname{Soc}(M)} \simeq H,$$

it is now sufficient to show that $H$ is noetherian in order to obtain that $\frac{M}{\operatorname{Soc}(M)}$ is noetherian. Let $X = X_1 \oplus X_2 \oplus \cdots$ be a direct sum of nonzero submodules contained in $H$. Since for each $i$, $\operatorname{Soc}(M) \cap X_i = 0$, it follows that each $X_i$ contains a proper essential submodule $Y_i$. Thus $Y = Y_1 \oplus Y_2 \oplus \cdots$ is an essential submodule of $X$, and therefore $\frac{X}{Y} \simeq \frac{X_1}{Y_1} \oplus \frac{X_2}{Y_2} \oplus \cdots$ is noetherian. Consequently, the sum $X = X_1 \oplus X_2 \oplus \cdots$ must be a finite sum, and therefore $H$ has finite Goldie dimension, say, $k$. Let $U = U_1 \oplus \cdots \oplus U_k$ be a direct sum of uniform submodules which is essential in $H$. Since $H/U$ and $U$ are noetherian, $H$ is noetherian. This proves the claim. The converse is trivial.

**Lemma 4.** *Let $M = \bigoplus \sum_{i=1}^{n} A_i$. Then $M$ is continuous if and only if each $A_i$ is continuous and $A_j$-injective for $j \neq i$.*

*Proof.* This is a special case of [4, Theorem 13].

## 3. THE RESULT

**Theorem.** *Let $R$ be a ring with ascending chain condition on essential left ideals.*

(i) *If $R$ is left or right continuous, then $R$ is semiperfect.*
(ii) *If $R$ is left continuous, then $R$ is left artinian.*
(iii) *If $R$ is right continuous, then $R$ need not be left artinian.*

*Proof.* (i) By Lemma 2, $\overline{R} = R/J(R)$ is von Neumann regular and left (right) continuous. Because $\overline{R}$ is regular, the right socle of $\overline{R} =$ left socle of $\overline{R} = S$. Since $\overline{R}$ also satisfies the acc on essential left ideals, it follows from Lemma

3 that $Q = \overline{R}/S$ is semisimple artinian. Let $M$ be a singular left (right) $\overline{R}$-module. Since $SM = 0 (MS = 0)$, $M$ is a $Q$-module and therefore ${}_Q M (M_Q)$ is injective. This implies that $M$ is injective as an $\overline{R}$-module because $\overline{R}$ is regular. Next we show that every cyclic left (right) $\overline{R}$-module is continuous. For, let $\overline{I}$ be a left (right) ideal of $\overline{R}$. Then $\overline{I}$ is essential in $\overline{A}$, where $\overline{R} = \overline{A} \oplus \overline{B}$, since $\overline{R}$ is continuous. Thus, $\overline{A}/\overline{I}$ is singular and hence injective as an $\overline{R}$-module. Also, by Lemma 4, $\overline{B}$ is $\overline{A}$-injective and so $\overline{B}$ is $\overline{A}/\overline{I}$-injective. Now from $\overline{R}/\overline{I} \simeq \overline{A}/\overline{I} \times \overline{B}$, it follows by invoking again Lemma 4 in the other direction that $\overline{R}/\overline{I}$ is continuous. Since a ring each of whose cyclic left (right) modules is continuous is always semiperfect ([3, p. 201]), the regular ring $\overline{R}$ must be semisimple artinian. Hence by Lemma 2, $R$ is semiperfect.

(ii) By Lemma 2 and Lemma 1, $R$ is semiprimary. Write $R = \oplus \sum_{i=1}^{k} \mathrm{Re}_i$ as a direct sum of indecomposable left ideals. Since each $\mathrm{Re}_i$ is continuous, $\mathrm{Soc}(\mathrm{Re}_i)$ is simple or zero and so $\mathrm{Soc}({}_R R)$ is finitely generated. Thus by Lemma 3, $R$ is left noetherian and hence $R$ is left artinian because $R$ is semiprimary.

(iii) Let $R$ be a ring with only three right ideals $0$, $J(R)$, and $R$, which is not left artinian (see [1, Example 7.11'.1, p. 337]). $R$ is clearly right continuous. Also, $R$ has acc on essential left ideals since $J(R) = \mathrm{Soc}({}_R R)$ (Lemma 3). Incidently, $R$ is not even left noetherian.

This completes the proof.

## 4. Remarks

(i) It follows from our theorem and that of Utumi, ([6, Theorem 7.10]) that a left and right continuous ring with acc on essential left and essential right ideals is quasi-Frobenius.

(ii) Contrary to the fact that for a right self-injective ring $R$ the minimum condition on one side implies the minimum condition on the other side and the left self-injectivity of $R$, the example in part (iii) of the theorem shows that a right continuous right artinian ring need not be left artinian or left continuous.

(iii) A two-sided artinian ring with one-sided continuity is not necessarily continuous on the other side:

Let $R$ be a ring with only three left ideals $0$, $J(R)$, and $R$, which is a right artinian with composition length 3 ([1, Example 7.11'.2, p. 338]). $R$ is clearly left continuous. However, $R$ is not right continuous. For otherwise, by Remark (i), $R$ would be quasi-Frobenius, and hence uniserial with composition length 2 (on either side).

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