Reprinted from JOURNAL OF ALGEBRA
All Rights Reserved by Academic Press, New York and London

Vol. 128, No. 1, January 1990
Printed in Belgium

208

Rings Whose Cyclics Are Essentially Embeddable in Projective Modules

S. K. JAIN AND S. R. LÓPEZ-PERMOUTH

Department of Mathematics, Ohio University, Athens, Ohio 45701

Communicated by Barbara L. Osofsky

Received July 28, 1988

1. INTRODUCTION

A ring R is quasi-Frobenius (QF) if and only if every right R-module is embeddable in a projective module. We call a ring R a right (left) CEP-ring if each cyclic right (left) R-module is essentially embeddable in a projective module. Examples of right CEP-rings include QF-rings and right uniserial rings. Indeed R is a QF-ring if and only if R is both a right and left CEP-ring [6]. A right CEP-ring which is QF-3 is shown to be QF (Theorem 3.3). Semiperfect CEP-rings and rings, each of whose homomorphic images is a right CEP-ring, are characterized in Theorems 5.2 and 6.2, respectively. The last section deals with split extensions of right uniserial rings as examples of CEP-rings.

2. DEFINITIONS AND NOTATION

A ring R is called right (left) uniserial if and only if it has a unique finite composition series of right (left) ideals. R is uniserial if it is both right and left uniserial. R is known to be uniserial if and only if R is right (left) uniserial and either right or left self-injective. A ring R is called a (right) *QF*-3 ring if its injective hull as a right R-module is projective.

Let M and N be right (left) R-modules. M is called weakly N-injective if for each $\sigma \in \mathrm{Hom}(N, E(M))$, where $E(M)$ is an injective hull of M, there exists a submodule X of $E(M)$ such that $\sigma(N) \subset X \simeq M$. Thus, if M is N-injective then $\sigma(N)$ is isomorphic to a submodule of M for each $\sigma \in \mathrm{Hom}(N, E(M))$. Clearly, every N-injective module is weakly N-injective.

J or $\mathrm{Rad}(R)$ will denote the Jacobson radical of R. $A \subset' B$ ($A \subsetneq' B$) will denote that A is essential (essentially embeddable) in B. $E(X)$ will denote

0021-8693/90 $3.00

the injective hull of a module X. The right (left) annihilator of X in S will be denoted by $\text{r.ann}_S(X)$ ($\text{l.ann}_S(X)$). We write $\text{r.}Q_{cl}(R)$ ($\text{l.}Q_{cl}(R)$) to denote the right (left) classical ring of quotients of R. Utumi's ring of right quotients of R will be denoted by $Q(R)$. As usual mod-R (R-mod) denotes the category of right (left) R-modules. Throughout our paper, unless otherwise stated, all modules are right unital and by a CEP-ring we mean a right CEP-ring.

3. Preliminary Results

In this section we show that a semiperfect CEP-ring is right artinian and that all its projective indecomposable right ideals (hence all projective indecomposable modules) are uniform (Theorem 3.2). Write $R = \oplus \sum_{i=1}^{n} e_i R$, where $e_i e_j = \delta_{ij} e_i$, $1 = \sum_{i=1}^{n} e_i$, and $e_i R$, $1 \leqslant i \leqslant n$, are indecomposable right ideals. After renumbering, if necessary, let $e_1 R, ..., e_k R$ be a complete set of projective indecomposable right ideals. Throughout this paper, unless otherwise stated, we shall represent a semiperfect ring R as $\oplus \sum_{i=1}^{n} e_i R$, where the e_i's are as described above.

3.1. Lemma. *If R is a semiperfect CEP-ring and P is a projective module then* $\text{Soc}(P) \subset' P$. *Furthermore, if Q is another projective module such that* $\text{Soc}(Q) \simeq \text{Soc}(P)$ *then* $Q \simeq P$.

Proof. Since R is semiperfect, we may write $R = e_1 R \oplus \cdots \oplus e_n R$, where $\mathcal{P} = \{e_1 R, ..., e_k R\}$ $(k \leqslant n)$ is an irredundant complete set of representatives for the projective indecomposable R-modules. Let $\mathcal{S} = \{S_1, ..., S_k\}$ be an irredundant complete set of representatives for the simple R-modules. Since every simple module S_i is cyclic, S_i is essentially embeddable in some projective module P which must be indecomposable (and hence $P \simeq e_j R$ for some j). Thus we can define a function $f: \mathcal{S} \to \mathcal{P}$ by $f(S_i) = e_j R$. The function f must be one-to-one, hence onto. Thus for any indecomposable projective module $e_i R$, $\text{Soc}(e_i R)$ is the unique simple essential submodule, proving the statement of the lemma for indecomposable projectives. Let P be an arbitrary projective module. Since R is semiperfect, there exist sets A_i, $i = 1, ..., k$, such that $P \simeq \oplus \sum_{i=1}^{n} (e_i R)^{(Ai)}$. Since $\oplus \sum (\text{Soc}(e_i R))^{(Ai)} \subset' \oplus \sum (e_i R)^{(Ai)}$, it follows that $\text{Soc}(P) \subset' P$. Suppose $Q = \oplus \sum_{i=1}^{k} (e_i R)^{(Bi)}$ is such that $\text{Soc}(Q) \simeq \text{Soc}(P)$. Then

$$\oplus \sum_{i=1}^{k} (\text{Soc}(e_i R))^{(Bi)} \simeq \oplus \sum_{i=1}^{k} (\text{Soc}(e_i R))^{(Ai)},$$

and so by the Krull–Schmidt theorem there is a bijection between A_i and B_i, for $i = 1, ..., k$. Therefore, $P \simeq Q$. ∎

3.2. LEMMA. *A semiperfect right CEP-ring is right artinian. All projective indecomposable right modules over a semiperfect right CEP-ring are uniform.*

Proof. Since each cyclic R-module is essentially embeddable in a projective module, it folows from Lemma 3.1 that each cyclic R-module has nonzero socle. Thus R is left perfect. Furthermore, since $J(R)/(J(R))^2$ is completely reducible and hence embeddable in $\mathrm{Soc}(R^m)$, for some m, $J(R)/(J(R))^2$ is finitely generated and so R is right artinian [1, p. 322). The statement concerning indecomposable projectives was proved in the course of the proof of Lemma 3.1. It is stated here for future reference. ∎

As a consequence of the above results we obtain the following characterization of QF-rings.

3.3. THEOREM. *For an arbitrary ring R, the following are equivalent*:

(1) *R is QF.*

(2) *R is CEP and $QF-3$.*

(3) *Every cyclic R module has a projective injective hull.*

Proof. The implications $(1)\Rightarrow(3)\Rightarrow(2)$ are clear. To prove $(2)\Rightarrow(1)$, let $E(R)$ be projective. Then $E(R)$ and R are projective modules with isomorphic socles and therefore, by Lemma 3.1, $E(R)\simeq R$. ∎

4. WEAK RELATIVE INJECTIVITY

Let M and N be R-modules. Recall that M is weakly N injective if and only if for every $\sigma\in \mathrm{Hom}(N, E(M))$ there exists $X\subset E(M)$ such that $X\simeq M$ and $\sigma(N)\subset X$.

4.1. PROPOSITION. *The following statements are equivalent*:

(i) *M is weakly N-injective.*

(ii) *For every submodule L of N and every monomorphism $\sigma: N/L\to E(M)$, there exists $X\subset E(M)$ such that $X\simeq M$ and $\sigma(N/L)\subset X$.*

(iii) *For every submodule L of N, M is weakly N/L-injective.*

Proof. The proof folows immediately from the definition of weak relative injectivity. ∎

For the special case $M=N=R$ we have the following useful characterization.

4.2. PROPOSITION. *R is weakly R-injective if and only if for all $a \in E(R)$ there exists an element $b \in E(R)$ such that* $\mathrm{r.ann}_R(b) = 0$ *and $a \in bR$.*

Proof. Let $\sigma: R \to E(R)$ be the homomorphism defined by $\sigma(r) = ar$, $r \in R$. Then $aR = \sigma(R)$ is contained in a submodule X of $E(R)$ which is isomorphic to R via $\varphi: R \to X$. Let $b = \varphi(1)$, Then $a \in bR$ and $\mathrm{r.ann}_R(b) = 0$. ∎

Weak relative injectivity is closed under finite direct sums (Lemma 4.3) but the direct summands of a weakly injective module need not be weakly injective (see Example 4.4(d)).

The following lemma is immediate but we record it for the sake of reference.

4.3 LEMMA. *If L and M are weakly N-injective modules, then $L \oplus M$ is weakly N-injective.*

Proof. Straightforward. ∎

4.4. EXAMPLES. (a) Every right Ore-domain R is weakly R-injective.

Since $E(R) = \mathrm{r.}Q_{\mathrm{cl}}(R)$ is a division ring, the statement follows from Proposition 4.2.

(b) Every prime (right and left) noetherian ring is weakly R-injective.

Here $E(R_R) = \mathrm{l.}Q_{\mathrm{cl}}(R) = Q$. So if $q \in E(R)$ then there exists an essential left ideal K such that $Kq \subset R$. But since K contains a unit in Q, it follows by Proposition 4.2 that R is weakly R-injective.

(c) A Boolean ring R is weakly R-injective if and only if it is injective. For, if R is Boolean then $E(R) = Q(R)$ is again a Boolean ring. So if R is weakly R-injective and $q \in Q(R)$, there exists a regular element $b \in Q(R)$ such that $q \in bR$. But since the only regular element in a Boolean ring is 1, we conclude $R = Q(R)$.

In particular, the ring R of all finite or cofinite sets of natural numbers under the usual Boolean operations is not weakly R-injective.

We now give an example that a direct summand of a weakly R-injective module need not be weakly R-injective.

(d) Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a field. We show that R is weakly R-injective but $e_{22}R$ is not weakly R-injective, where $e_{22} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Proof. Here $E(R) = \left(\begin{smallmatrix} F & F \\ F & F \end{smallmatrix}\right)$. So let $q = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in E(R)$. By Proposition 4.2, we need to show that there exists an invertible element $q' \in E(R)$ such that $q'q \in R$. Without loss of generality, we may assume $c \neq 0$. Choose

$$q' = \begin{cases} \begin{pmatrix} a & 0 \\ -c & a \end{pmatrix}, & \text{if } a \neq 0 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{if } a = 0. \end{cases}$$

Then q' is invertible in $E(R)$ and $q'q \in R$.

To show $e_{22}R$ is not weakly R-injective, we consider the R-homomorphism $\sigma: R \to E(e_{22}R) = \left(\begin{smallmatrix} 0 & 0 \\ F & F \end{smallmatrix}\right)$ given by $\sigma\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Clearly, σ is onto. So $\sigma(R) = \left(\begin{smallmatrix} 0 & 0 \\ F & F \end{smallmatrix}\right)$ is not embeddable in $e_{22}R$. Hence $e_{22}R$ is not weakly R-injective. ∎

We conclude this section with an important lemma.

4.5. Lemma. *Let R be a right artinian ring, and let M and N be finitely generated R-modules. If M is weakly N-injective, N is weakly M-injective, and* $\mathrm{Soc}(M) \simeq \mathrm{Soc}(N)$, *then* $M \simeq N$.

Proof. Consider the embedding

$$\sigma: \mathrm{Soc}(M) \to E(N)$$

induced by the isomorphism between $\mathrm{Soc}(M)$ and $\mathrm{Soc}(N)$. Now σ can be extended to $\hat{\sigma}: M \to E(N)$ which is again a monomorphism. Since N is weakly M-injective, $\hat{\sigma}(M) \subset X$, $X \simeq N$; and so M is embeddable in N. Similarly, N is embeddable in M. Because M and N are finitely generated modules over a right artinian ring, it follows that $M \simeq N$. ∎

5. CEP-Rings

In this section we characterize the class of CEP-rings. We start with a key lemma.

5.1. Lemma. *Let R be a right artinian ring such that all indecomposable projective R-modules are uniform and weakly R-injective. Then*

(i) *every simple R-module is isomorphic to the socle of an indecomposable projective module,*

(ii) *every simple R-module is embeddable in* $\mathrm{Soc}(R)$, *and*

(iii) *if P and Q are projective modules with* $\mathrm{Soc}(P) \simeq \mathrm{Soc}(Q)$ *then* $P \simeq Q$.

213

Proof. Write $R = \oplus \sum_{i=1}^{n} e_i R$ as a direct sum of indecomposable right ideals, where $\mathscr{P} = \{e_1 R, ..., e_k R\}$ $(k < n)$ is an irredundant complete set of representatives for the projective indecomposable modules. Let $S_i = \mathrm{Soc}(e_i R)$. Clearly $e_i R \simeq e_j R$ implies $S_i \simeq S_j$. By Lemma 4.5 if $S_i \simeq S_j$ then $e_i R \simeq e_j R$. Since $\mathscr{S} = \{S_1, S_2, ..., S_k\}$ is an irredundant set of simple R-modules containing $k = |\mathscr{P}|$ members, $\mathscr{S}$ must be a complete set of representatives.

This proves (i) and (ii), and also (iii) when P and Q are indecomposable. In general,

$$P \simeq \oplus \sum_{i=1}^{k} (e_i R)^{(A_i)} \qquad \text{and} \qquad Q \simeq \oplus \sum_{i=1}^{k} (e_i R)^{(B_i)}.$$

So $\mathrm{Soc}(P) \simeq \mathrm{Soc}(Q)$ yields $|A_i| = |B_i|$, proving $P \simeq Q$. ∎

5.2. THEOREM. *A semiperfect ring R is CEP if and only if the following hold*:

(i) *R is right artinian,*

(ii) *every indecomposable projective module is uniform, and*

(iii) *Every indecomposable projective module is weakly R-injective.*

Proof. Let R be a semiperfect CEP-ring. Then R satisfies (i) and (ii) (Theorem 3.2). Let eR be an indecomposable projective. By (ii), $\mathrm{Soc}(eR)$ is simple. Let $\sigma: R/I \to E(eR)$ be a monomorphism. Then $\mathrm{Soc}(R/I) \simeq \mathrm{Soc}(E(eR)) = \mathrm{Soc}(eR)$. Since R is a CEP-ring, $R/I \subsetneq' P$, for some projective module P. Therefore $\mathrm{Soc}(P) \simeq \mathrm{Soc}(R/I) \simeq \mathrm{Soc}(eR)$. Hence, by Lemma 3.1, $P \simeq eR$ and so there exists a map φ: $R/I \to eR$ which embeds R/I essentially in eR. Then, there exists $\hat{\sigma}: eR \to eR$ which embeds R/I essentially in eR. Then, there exists $\hat{\sigma}: eR \to E(eR)$ such that $\hat{\sigma} \circ \varphi = \sigma$. Clearly, $\hat{\sigma}$ is one-to-one. Let $X = \hat{\sigma}(eR)$. Then $\sigma(R/I) \subset X$ and $X \simeq eR$. Conversely, assume R satisfies (i)–(iii). Write $R = \oplus \sum_{i=1}^{n} e_i R$ as a direct sum of indecomposable right ideals, where $\mathscr{P} = \{e_1 R, ..., e_k R\}$ $(k \leqslant n)$ is an irredundant complete set of representatives for the projective indecomposable R-modules. Let I be a right ideal of R. Now by Lemma 5.1(i), $\mathrm{Soc}(R/I) \simeq \oplus \sum_{i=1}^{k} (\mathrm{Soc}(e_i R))^{n_i}$, $n_i \geqslant 0$, $i = i, ..., k$.

Let $P = \oplus \sum_{i=1}^{k} (e_i R)^{n_i}$. Since $\mathrm{Soc}(P) = \mathrm{Soc}(E(P))$, the above isomorphism between $\mathrm{Soc}(R/I)$ and $\oplus \sum_{i=1}^{k} (\mathrm{Soc}(e_i R))^{n_i} = \mathrm{Soc}(P)$ may be looked upon as an essential embedding φ: $\mathrm{Soc}(R/I) \to E(P)$. Extend φ to $\hat{\varphi}$: $R/I \to E(P)$. Clearly, $\hat{\varphi}$ is also one-to-one. Now by (iii), P is weakly R-injective. So P is also R/I-injective (Proposition 4.1). Therefore, there exists $X \subset E(P)$ such that $X \simeq P$ and $\hat{\varphi}(R/I) \subset X$. Now $X \supset \hat{\varphi}(R/I) \supset$

$(\mathrm{Soc}(R/I)) = \mathrm{Soc}(E(P))$. Therefore, $X \subset' E(P)$ and $\mathrm{Soc}(X) = \mathrm{Soc}(E(P))$. It follows then $X \supset \hat{\varphi}(R/I) \supset \mathrm{Soc}(X)$, and so $\hat{\varphi}(R/I) \subset' X$ as desired. ∎

5.3. *Remark.* Note that the condition (iii) in Theorem 4.2 may not be weakened to the more appealing condition "R is weakly R-injective" since weak relative injectivity is not inherited by direct summands. Example 4.4(d) provides a ring R satisfying (i) and (ii) which is weakly R-injective but which is not a CEP-ring.

6. Rings Whose Every Homomorphic Image is a CEP-Ring

In this section we characterize rings R satisfying the property that every homomorphic image of R is a CEP-ring. These rings turn out to be precisely those semiperfect rings R for which every cyclic R-module is embeddable essentially in a direct summand of R [7]. The fact that R is semiperfect follows as below. Since R/N is a CEP-ring, where N is the prime radical of R, every right ideal of R/N is an annihilator right ideal and hence R is semiperfect [3, p. 204, Exercise 24.3(d)].

6.1. Lemma. *Let R be a ring such that every homomorphic image is a CEP-ring. Then every indecomposable projective R-module is uniserial and all its composition factors are isomorphic to one another.*

Proof. Let m be a positive integer. Let $P(m)$ denote the statement that for any ring R each of whose homomorphic images is a CEP-ring, and for any indecomposable projective R-module P of composition length m, the following are true:

(i) P is uniserial, and

(ii) the composition factors of P are isomorphic to one another.

We proceed inductively to show that $P(m)$ is true for all m. $P(1)$ is clearly true. Let us assume that $P(m)$ holds for some m. Let R be a ring each of whose homomorphic images is a CEP-ring and let P be a projective indecomposable R-module of composition length $m+1$. Since R is semiperfect, we may write $R = e_1R \oplus \cdots \oplus e_nR$, where for some $t \leqslant n$, $P \simeq e_1R \simeq e_2R \simeq \cdots \simeq e_tR$ and $e_1R \not\simeq e_jR$ whenever $j > t$. Let $[e_1R] = e_1R \oplus \cdots \oplus e_tR$, and let $I = \mathrm{Soc}([e_1R]) = \mathrm{Soc}(e_1R) \oplus \cdots \oplus \mathrm{Soc}(e_tR)$. Then, if we define $S = R/I$,

$$S \simeq \frac{e_1R}{\mathrm{Soc}(e_1R)} \oplus \cdots \oplus \frac{e_tR}{\mathrm{Soc}(e_tR)} \oplus e_{t+1}R \oplus \cdots \oplus e_nR \qquad (1)$$

is a direct sum of indecomposable projective S-modules (Proposition 27.4 [1]). Since $(e_1 R)/(\mathrm{Soc}(e_1 R))$ is an indecomposable projective S-module with composition length m and S is a ring each of whose homomorphic images is a CEP-ring, it follows by $P(m)$ that $(e_1 R)/(\mathrm{Soc}(e_1 R))$ is uniserial as an S-module (and equivalently as an R-module). Because $\mathrm{Soc}(e_1 R)$ is simple, we obtain that $e_1 R$ is uniserial. Also, by $P(m)$, all composition factors of $(e_1 R)/(\mathrm{Soc}(e_1 R))$ are isomorphic to one another as S-modules and hence as R-modules. To complete our proof of $P(m+1)$, we need only to show that $\mathrm{Soc}(e_1 R/\mathrm{Soc}(e_1 R)) \simeq \mathrm{Soc}(e_1 R)$. Let k be the number of distinct isomorphism classes of R- (or S-) simple modules. We know $\mathrm{Soc}(e_t R + \cdots + e_n R)$ contains exactly $k-1$ nonisomorphic simple modules. Thus, by (1), the missing simple S-module $\mathrm{Soc}(e_1 R)$ must be embeddable in $\mathrm{Soc}(e_1 R/\mathrm{Soc}(e_1 R) \oplus \cdots \oplus e_t R/\mathrm{Soc}(e_t R))$ (Lemma 5.1(ii)). Hence, $\mathrm{Soc}(e_1 R) \simeq \mathrm{Soc}(e_1 R/\mathrm{Soc}(e_1 R))$. This proves $P(m+1)$ and concludes our induction. ∎

For convenience, we shall say that a right uniserial module M is homogeneous if all of its composition factors are isomorphic to one another.

6.2. *Theorem.* *For a ring R, the following are equivalent*:

(i) *Every homomorphic image of R is a CEP-ring.*

(ii) *R is of one of the following types*:

(a) *a right uniserial ring,*

(b) *an $n \times n$ matrix ring over a right self-injective right uniserial ring, $n > 1$, or*

(c) *a direct sum of rings of types (a) or (b).*

(iii) *Every cyclic R-module is essentially embeddable in a direct summand of R, and R is semiperfect.*

(iv) *Every ring homomorphic image S of R has the property that each cyclic S-module is essentially embeddable in a direct summand of S.*

Proof. The equivalence of (ii), (iii) and (iv) is shown in [7]. Trivially, (iv) implies (i). We now proceed to show that (i) implies (ii). By Lemma 5.1, $R = \bigoplus \sum_{i=1}^{n} e_i R$, where each $e_i R$ is a homogeneous uniserial module. Let $\sigma: e_i R \to e_j R$ be a nonzero homomorphism. Then $e_i R/\ker\sigma \subsetneq e_j R$ and so $e_i R$ and $e_j R$ have a common composition factor. But since both $e_i R$ and $e_j R$ are homogeneous, $e_i R/e_i J \simeq e_j R/e_j J$, yielding $e_i R \simeq e_j R$. This implies that we can rewrite $R = [e_1 R] \oplus \cdots \oplus [e_k R]$ as a direct sum of ideals $[e_i R]$, where $[e_i R] = \bigoplus \sum_s e_s R$, where the sum runs over all s such that

$e_s R \simeq e_i R$. Then, $R \simeq \bigoplus \sum_{i=1}^{k} M_{n_i}(e_i R e_i)$, where n_i is the number of direct summands in $[e_i R]$. Since $e_i R$ is uniserial, $e_i R e_i$ is a right uniserial ring. If $n_i > 1$, then it can be shown that $e_i R e_i$ is a right self-injective ring (see Lemma 6.3), completing the proof. ∎

We prove below that if an $n \times n$, $n > 1$, matrix ring R over a right uniserial ring S is a CEP-ring then S is right self-injective. The argument used here is the same as the argument used to prove a similar result for rings R each of whose cyclic R-modules is essentially embeddable in a direct summand of R [7]. We give the proof here for the sake of completeness.

6.3. LEMMA. *If R is the $n \times n$ matrix ring over a right uniserial ring S with $n > 1$, and R is a CEP-ring, then S must be right self-injective.*

Proof. The S-module S^n corresponds to the R-module R under the category isomorphism $- \otimes_R Re_{11}$: mod-$R \to$ mod-S.

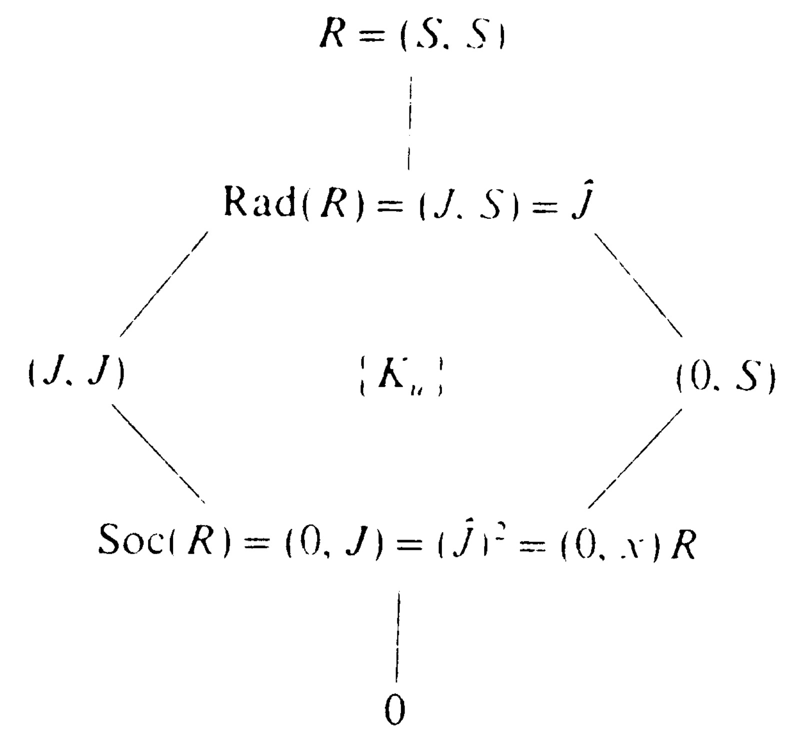
Since R is a CEP-ring, every quotient of S^n is embeddable in S^m, for some positive integer m. Let us write $\text{Rad}(S) = xS = J$. Assume that S is not right self-injective. Then, there exists an element $s \in S$, $s \notin J$, satisfying $xs \notin Sx$. Let $N = (x, -xs, 0, 0, ..., 0)\, S \subset S^n$. We claim that $(S^n)/N$ is not embeddable in S^m, for any m. Let $\bar{e}_i = e_i + N$, $i = 1, 2$, where $e_1 = (1, 0, ..., 0)$ and $e_2 = (0, 1, 0, ..., 0)$. Then $\bar{e}_1 S$ and $\bar{e}_2 S$ are both isomorphic to S. Also, $\bar{e}_1 S \cap \bar{e}_2 S = \bar{e}_1 xS = \bar{e}_2 xS$. If $\psi: S^n/N \to S^m$ were an embedding of S^n/N into S^m, with $\psi(\bar{e}_1) = (a_1, a_2, ..., a_m)$ and $\psi(\bar{e}_2) = (b_1, b_2, ..., b_m)$, then there must exist i, j such that a_i and b_j are invertible. However, $\psi(\bar{e}_1 x) = (a_1 x, a_2 x, ..., a_m x)$, and $\psi(\bar{e}_2 xs) = (b_1 xs, b_2 xs, ..., b_m xs)$. This implies that $a_j x = b_j xs$ and therefore $b_j^{-1} a_j x = xs$, contradicting our choice of s. Therefore, our assumption that S is not right self-injective does not hold. ∎

7. EXAMPLES

In this section we provide examples to illustrate the concepts developed in this paper.

7.1. EXAMPLE. Our first example is a local CEP-ring which is neither right uniserial nor quasi-Frobenius. This also provides us with an example of a CEP-ring not all of whose homomorphic images are CEP-rings.

Let S be a ring having only three right ideals, namely, S, $J = \text{Rad}(S) = xS$ and (0), which is not necessarily right self-injective [2, p. 337]. Let $R = (S, S)$ denote the split extension of S, i.e., $R = \{(a, b) \mid a, b \in S\}$ with componentwise addition and multiplication given by $(a, b)(c, d) = (ac, ad + bc)$. The lattice of right ideals of R is



where $K_u = (x, u)R$, $u \notin J$. We note that

(i) $R/\hat{J} \simeq (0, J) \subset' R$,

(ii) $R/(0, S) \simeq (0, S) \subset' R$,

(iii) $R/(J, J) \simeq (J, J) \subset' R$,

(iv) $R/(0, J) \subset' R \times R$, since $(0, J) = (0, x)R = \mathrm{r.ann}_R((0, 1), (x, 0))$,

and

(v) $R/K_u \simeq K_{-u} \subsetneq' R$, under the map sending $1 + K_u$ into $(xu^{-1}, -1)$.

This shows that R is a CEP-ring, R is not right uniserial, and R is not right self-injective whenever S is not right self-injective [4]. Finally, $R/\mathrm{Soc}(R)$ is not a CEP-ring since it is local but not uniform.

Our next example shows that the split extension of a right uniserial ring of composition length greater than 2 need not be a CEP-ring. A necessary and sufficient condition is given for the split extension of a right uniserial ring of composition length 3 to be a CEP-ring.

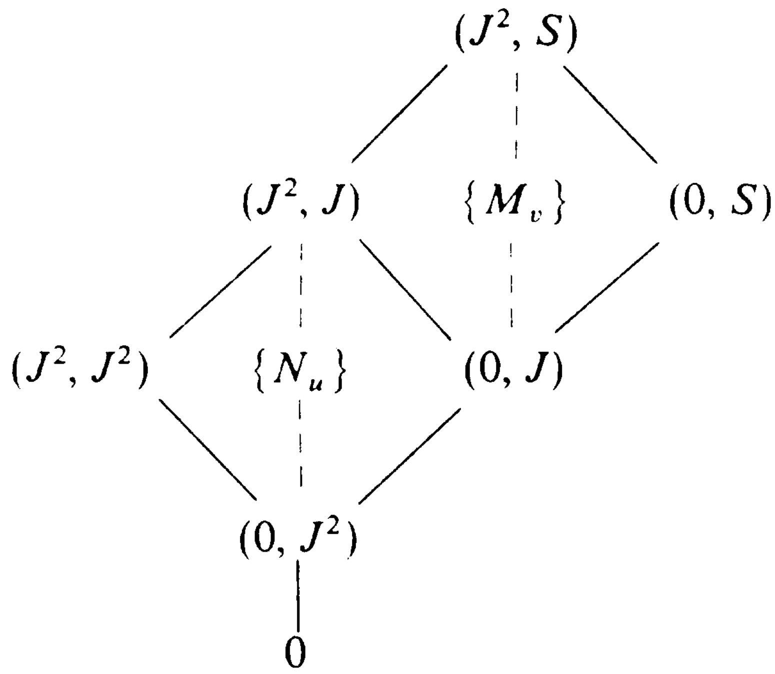
7.2. Example. Let S be a right uniserial ring with $\mathrm{Rad}(S) = xS = J$ such that $J^3 = 0 \neq J^2$. Let R be a split extension of S as defined in Example 7.1. Notice that the ring $R/(J^2, J^2)$ is isomorphic to the split extension of S/J^2 which is a right uniserial ring of composition length 2. Therefore, if K

is a right ideal of R containing (J^2, J^2), R/K is embeddable essentially in $R/(J^2, J^2)$ of $R/(J^2, J^2) \times R/(J^2, J^2)$, as shown in Example 7.1. Since $R/(J^2, J^2) \simeq (J, J) \subset' R$, $R/K \subsetneq' R^{(i)}$ for $i = 1$ or 2, for all K containing (J^2, J^2). Next suppose K does not contain (J^2, J^2). Then we have the following cases.

(i) For $K = (0, J^i)$, $i = 1, 2$, $K = \text{r. ann}_R((0, 1), (X^{3-i}, 0))$. So $R/K \subset R \times R$. Further, $\text{Soc}(R \times R) = (0, J^{n-1}) \times (0, J^{n-1})$ while $\text{Soc}(R/K)$ contains at least two direct summands $(J^{3-i}, J^i)/K$ and $(0, J^{i-1})/K$. Therefore, $R/K \subsetneq' R \times R$.

(ii) If $K = (0, S) = \text{r.ann}_R(0, 1)$ then $R/K \subset' R$.

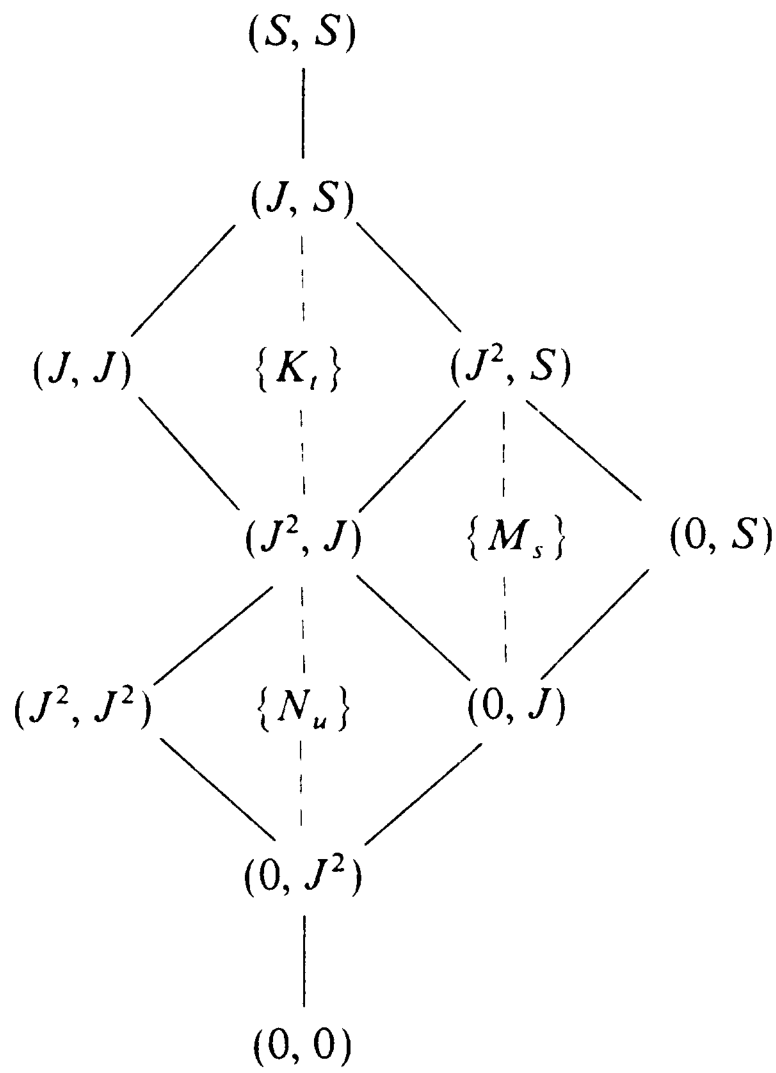
(iii) Let us now assume $K \neq (0, J^i)$ ($i = 0, 1$, or 2) and K does not contain (J^2, J^2). It can be shown that K must be of the form N_u or M_v as in the diagram



where $N_u = (x^2, xu) R$, and $M_v = (x^2, v) R$, u and v invertible. Then $R/M_v \simeq M_v$, via the map which sends $1 + M_v$ to $(x^2 v^{-1}, -1)$. Therefore, R is a CEP-ring if and only if, for every invertible element u of S, the cyclic module R/N_u is embeddable essentially in a projective R-module. We show next (Remark 7.3) that this is possible if and only if, for every invertible $u \in S$, there exist v and $w \in S$ (necessarily invertible) such that $xvxu = wx^2$.

7.3. *Remark.* Let S be a right uniserial ring with $\text{Rad}(S) = xS = J$ such that $J^3 = (0) \neq J^2$. Let R be the split extension of S. Then R is a CEP-ring if and only if for every invertible $u \in S$ there exist invertible elements v, $w \in S$ such that $xvxu = wx^2$.

Proof. As indicated in Example 7.2 the lattice of right ideals of R is



where

$$K_t = (x, t)\,R$$
$$M_s = (x^2, s)\,R$$
$$N_u = (x^2, xu)\,R$$
$$t, s, u \in S,\ t, s, u \notin J.$$

Furthermore, as remarked towards the end of Example 7.2, R is a CEP-ring if and only if R/N_u is essentially embeddable in a projective module for each invertible $u \in S$. Now $\mathrm{Soc}(R/N_u)$ is simple. So if R/N_u were essentially embeddable in a projective module, then $P \simeq R$. Furthermore, the composition length of R/N_u is 4. Thus $R/N_u \subsetneq' R$ if and only if

(i) $R/N_u \simeq (J, J)$, (ii) $R/N_u \simeq (J^2, S)$, or (iii) $R/N_u \simeq K_t$, for some $t \in S$, $t \notin J$.

We first rule out the possibilities (i) and (ii). Since $(J, J)(x^2, 0) = (0, 0)$, but $(x^2, 0) \notin N_u$, it follows that $R/N_u \not\simeq (J, J)$. Next, suppose that $\varphi\colon R/N_u \to (J^2, S)$ is an isomorphism with $\varphi(1 + N_u) = (x^2 a, b)$. Note that $a \notin J$, and $N_u = \mathrm{r.ann}_R(x^2 a, b)$. In particular, $(x^2 a, b)(x^2, xu) = (0, 0)$, which implies $b \in J$. This yields $\varphi(R/N_u) \subset (J^2, J)$, a contradiction.

We now proceed to prove that (iii) is true if and only if there exist invertible elements $v, w \in S$ such that $xvxu = wx^2$. Assume (iii) and let $\varphi: R/N_u \to K_t$ be an isomorphism. Let $\varphi(1 + N_u) = (xa, b)$. Since $(xa, b)R = (x, t)R$, a does not belong to J. Furthermore, $b \notin J$; for otherwise $(xa, b)R \subset (J, J)$. Choose $v = a$ and $w = -b$. Then $N_u = \text{r.ann}_R(xv, -w)$ and so $(xv, -w)(x^2, xu) = (0, xvxu - wx^2) = (0, 0)$. Therefore, $xvxu = wx^2$ as desired. Conversely, let $xvxu = wx^2$ for some invertible elements $v, w \in S$. Then we assert $N_u = \text{r.ann}_R(xv, -w)$. Clearly, $N_u \subset \text{r.ann}_R(xv, -w)$. So, let $(xv, -w)(y, z) = 0$. Then $xvy = 0 = xvz - wy$. Now $xvy = 0$ implies $y \in J^2$, and $xvz = wy$ implies $z \in J$. If $y = 0$ then $z \in J^2$ and so $(y, z) \in (0, J^2) \subset N_u$. If $y \neq 0$ then $y = x^2\alpha$, where α is invertible. Write $z = x\beta$. Then β must also be invertible. This transforms $xvz - wy = 0$ to $xvx\beta - wx^2\alpha = 0$. Since, by hypothesis, $xvxu = wx^2$, we obtain that $xvx(\beta - u\alpha) = 0$. Therefore, $\beta - u\alpha \in \text{r.ann}_R(xvx) \subset J$. Write $\beta - u\alpha = x\gamma$. Then $y = x\beta = x^2\gamma + xu\alpha$ and so $(x, y) = (x^2, xu)(\alpha, \gamma) \in N_u$ as desired. This completes the proof. ∎

We conclude by pointing out that the above remark guarantees that R is a CEP-ring if $x^2S = Sx^2$ (in particular, as is well known [4], if S is right self-injective).

Note added in proof. (1) We have learned after the submission of this paper that P. Menal has given an alternative proof of Theorem 3.3. in "On the Endomorphism Ring of a Free Module," *Publ. Secc. Mat. Univ. Autonoma de Barcelona* **27** (1985), 141–154. (2) We can prove now that conditions (i) and (iii) in Theorem 5.2 imply (ii).

References

1. F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," Springer-Verlag, New York/Heidelberg/Berlin, 1974.
2. C. Faith, "Algebra: Rings, Modules and Categories I," Springer-Verlag, New York/Berlin/Heidelberg, 1973.
3. C. Faith, "Algebra: Rings, Modules and Categories II," Springer-Verlag, New York/Berlin/Heidelberg, 1976.
4. C. Faith, Self-injective rings, *Proc. Amer. Math. Soc.* **77** (1979), 157–164.
5. C. Faith, Embedding modules in projectives: A report on a problem, *in* "Advances in Non-commutative Ring Theory," Vol. 951, Lectures Notes in Mathematics, Springer-Verlag, New York/Berlin/Heidelberg, 1981.
6. C. Faith and E. A. Walker, Direct sum representation of injective modules, *J. Algebras* **5** (1967), 203–221.
7. S. K. Jain and S. R. López-Permouth, A generalization of the Wedderburn–Artin theorem, *Proc. Amer. Math. Soc.* **106** (1987), 19–23.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium