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WEAKLY PROJECTIVE MODULES

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ABSTRACT

A module M is said to be weakly projective if and only if it has a projective cover $\pi : P(M) \rightarrow M$ and every map from $P(M)$ into a finitely generated (free) module can be factored through M via an epimorphism (not necessarily equal to π). In this paper we investigate the basic properties of weakly projective modules. These properties are dual to those known for weakly injective modules. In particular, we show that over a right perfect ring R there exists a right module K such that for any other module M , the direct sum $M \oplus K$ is weakly projective.

1. Introduction

The purpose of this paper is to study a concept dual to that of weak injectivity as in [1, 5, 6, 9, etc.]. Given a module M with projective cover $\pi : P(M) \rightarrow M$ and another module N , we say that M is weakly N -projective if and only if all maps from $P(M)$ into N factor through M via an epimorphism $\sigma : P(M) \rightarrow M$ (not necessarily equal to π). We dualize most of the basic results in [6], provide several examples of weakly projective modules which are not projective, and indeed show that, over a perfect ring R there exists a module K such that, for any other R -module M , the direct sum $M \oplus K$ is weakly projective.

We assume all modules are right and unital unless otherwise indicated. Any terminology used but not defined in this paper will be standard unless a specific reference is given. Sources for standard terminology include [2, 3 and 8]. A submodule $N \subset M$ is said to be a small submodule (denoted $N \ll M$) if the only submodule $K \subset M$ such that $K + N = M$ is $K = M$. Given a module M and a submodule $N \subset M$, a supplement of N in M is a submodule $K \subset M$ minimal with respect to the property that $N + K = M$. Equivalently, K is a supplement of N in M if and only if K satisfies both that $K + N = M$ and $K \cap N \ll M$. A module M is said

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to be hollow if every proper submodule of M is small in M or, equivalently, if every proper submodule of M has supplement in M equal to M . Unlike the dual concept of the complement of a submodule, not all submodules of a given module need have supplements. A module all of whose submodules have a supplement is said to be a supplemented module. Over a right perfect ring, all right modules are supplemented. A superfluous cover of a module M is a module N together with an epimorphism $p : N \rightarrow M$ such that $\text{Ker } p$ is small in N . Equivalently, one may think of a superfluous cover for M as being a module N such that $N/K \cong M$ for some small submodule $K \subset N$. A projective superfluous cover will be referred to, as is customary, a projective cover.

For any submodule K of a module N the natural inclusion map will be denoted by $i_K : K \rightarrow N$ and the natural projection by $\pi_K : N \rightarrow N/K$.

2. Basic Definitions and Results

Given modules M and N we say that M is N -projective if and only if every homomorphism $f : M \rightarrow N/K$ into a homomorphic image of N may be lifted to $\hat{f} : M \rightarrow N$ through the natural projection $\pi : N \rightarrow N/K$. When M has a projective cover one gets the following characterization of relative projectivity.

2.0 Theorem. Let M and N be modules and assume M has a projective cover P via an onto-homomorphism $\pi : P \rightarrow M$. Then M is N -projective if and only if for every homomorphism $\varphi : P \rightarrow N$ there exists a homomorphism $\hat{\varphi} : M \rightarrow N$ such that $\hat{\varphi}\pi = \varphi$. Equivalently, $\varphi(\text{ker } \pi) = 0$.

Proof. The proof is basically that of Proposition 2.2 in [10], once you realize that there is neither a need for $M = N$ nor for N to have a projective cover. However, for the sake of completeness (as suggested by the referee) we prove it as below.

Only if direction. Let $\varphi : P \rightarrow N$ be a homomorphism. We shall first show that $\varphi(\text{ker } \pi) = 0$. Let $T = \varphi(\text{ker } \pi)$ and let $\pi_T : N \rightarrow N/T$ be the natural projection. Then φ induces $\hat{\varphi} : M \rightarrow N/T$ defined by $\hat{\varphi}(m) = \pi_T \varphi(p)$, where $m = \pi(p)$. Clearly, $\hat{\varphi}\pi = \pi_T \varphi$. Since M is N -projective, there exists a map $\beta : M \rightarrow N$ such that $\hat{\varphi} = \pi_T \beta$. Clearly, $(\varphi - \beta\pi)P \subseteq T$. We claim that $\varphi = \beta\pi$.

Let $X = \{p \in P \mid \varphi(p) = \beta\pi(p)\}$. We shall show that $X = P$. Let $x \in P$. Since $(\varphi - \beta\pi)(x) \in T = \varphi(\text{ker } \pi)$, there exists $k \in \text{ker } \pi$ such that $(\varphi - \beta\pi)(x) = \varphi(k)$. Therefore, $\varphi(x - k) - \beta\pi(x - k) = 0$, since $\beta\pi(k) = 0$. Thus $x - k \in X$. Therefore, $\text{ker } \pi + X = P$, which implies $X = P$, since $\text{ker } \pi$ is small in P . Therefore, $(\varphi - \beta\pi)P = 0$. In particular, $(\varphi - \beta\pi)\text{ker } \pi = 0$, yielding $\varphi(\text{ker } \pi) = 0$. Equivalently, there exists $\varphi' : M \rightarrow N$ such that $\varphi' \pi = \varphi$.

Conversely, let $\psi : M \rightarrow N/K$ be a homomorphism. Then by the projectivity of P there exist a homomorphism $\psi' : P \rightarrow N$ such that $\psi\pi = \pi_K \psi'$. By our hypothesis there exists $\hat{\psi} : M \rightarrow N$ such that $\hat{\psi}\pi = \psi'$. It follows easily that $\pi_K \hat{\psi} = \psi$ as desired.

The above result is dual to a well-known characterization of relative-injectivity

and motivates the following definitions which are analogous to those of weak relative-injectivity and weak injectivity [5].

2.1 Definitions. Let M and N be modules and assume M has a projective cover $\pi : P \rightarrow M$. We say that M is weakly N -projective if for every map $\varphi : P \rightarrow N$ there exists an epimorphism $\sigma : P \rightarrow M$ and a homomorphism $\hat{\varphi} : M \rightarrow N$ such that $\varphi = \hat{\varphi}\sigma$. If a module M is weakly R^n -projective for all $n \in \mathbb{Z}^+$, we say that M is a *weakly projective* module.

The following characterization proves to be quite useful.

2.2 Theorem. Let M and N be modules and assume M has a projective cover $\pi : P \rightarrow M$. Then M is weakly N -projective if and only if for every map $\varphi : P \rightarrow N$ there exists a submodule $X \subset \text{Ker } \varphi$ such that $P/X \cong M$.

Proof. Let $\varphi : P \rightarrow N$ be a homomorphism. Assume first that M is weakly N -projective and let the homomorphism $\hat{\varphi} : M \rightarrow N$ and the epimorphism $\sigma : P \rightarrow M$ be as in the definition of weak relative-projectivity. Since $\varphi = \hat{\varphi}\sigma$, $\text{Ker } \sigma \subset \text{Ker } \varphi$. Also, $P/\text{Ker } \sigma \cong M$. Thus, the implication is proven by choosing $X = \text{Ker } \sigma$. Conversely, if $X \subset P$ satisfies the condition in the statement of the theorem, then the isomorphism $P/X \cong M$, composed with the natural projection $\pi_X : P \rightarrow P/X$ is an epimorphism $\sigma : P \rightarrow M$ satisfying that $\text{Ker } \sigma = X \subset \text{Ker } \varphi$. It follows that the map $\hat{\varphi} : M \rightarrow N$ given by $\hat{\varphi}(m) = \varphi(p)$, whenever $\sigma(p) = m$ is well defined and satisfies $\varphi = \hat{\varphi}\sigma$, proving our claim. \square

Domains of weak projectivity are closed under quotients and submodules, as shown in the next proposition.

2.3 Proposition. Let M and N be modules and assume M has a projective cover $\pi : P \rightarrow M$. Then the following statements are equivalent:

- (1) M is weakly N -projective,
- (2) for every submodule $K \subset N$, M is weakly K -projective, and
- (3) for every submodule $K \subset N$, M is weakly N/K -projective.

Proof. Since either condition (2) or (3) trivially implies (1), we need only show that (1) implies both (2) and (3). Assume M is weakly N -projective and let K be a submodule of N and $\varphi : P \rightarrow K$ be a homomorphism. Then $\Psi = i_K \varphi : P \rightarrow N$ may be expressed as a composition $\Psi = \hat{\Psi}\sigma$, for some homomorphism $\hat{\Psi} : M \rightarrow N$ and epimorphism $\sigma : P \rightarrow M$. Since σ is onto, the range of $\hat{\Psi}$ equals the range of Ψ and so it is contained in K . Thus, we may define $\hat{\varphi} : M \rightarrow K$ via $\hat{\varphi}(m) = \hat{\Psi}(m)$ and then $\varphi = \hat{\varphi}\sigma$, proving that M is weakly K -projective, as claimed. Assume once again that M is weakly N -projective and let $f : P \rightarrow N/K$ be a homomorphism. Since P is projective, there exists a map $g : P \rightarrow N$ such that $f = \pi_K g$. The weak N -projectivity of M yields an epimorphism $\sigma : P \rightarrow M$ and a homomorphism $\hat{g} : M \rightarrow N$ such that $g = \hat{g}\sigma$. Let $\hat{f} = \pi_K \hat{g}$. Then $\hat{f}\sigma = \pi_K \hat{g}\sigma = \pi_K g = f$, proving that M is indeed weakly N/K -projective.

While checking for weak projectivity one can actually restrict one's attention to epimorphisms, as follows.

2.4 Remark. Let M and N be modules and assume M has a projective cover $\pi : P \rightarrow M$. Then M is weakly N -projective if and only if for every submodule $K \subset N$ and for every epimorphism $\varphi : P \rightarrow K$ there exist epimorphisms $\sigma : P \rightarrow M$ and $\hat{\varphi} : M \rightarrow N$ such that $\varphi = \hat{\varphi}\sigma$.

Proof. Clear from the above proposition. □

One can obtain a dual for [Lemma 1.3, 6] in terms of supplements of submodules.

2.5 Proposition. Let M and N be modules and assume M is supplemented and has a projective cover $\pi : P \rightarrow M$. Then M is weakly N -projective if and only if for every submodule $K \subset N$ and for every epimorphism $\varphi : P \rightarrow K$ there exist an epimorphism $\hat{\varphi} : M \rightarrow K$ such that for every supplement L' of $\text{Ker } \hat{\varphi}$ in M there exists a submodule $L \subset P$ such that $P/L \cong M/L'$ and $L + \text{Ker } \varphi = P$.

Proof. Assume M is weakly N -projective and let $\varphi : P \rightarrow K$ be an epimorphism onto a submodule $K \subset N$. Then there exists epimorphisms $\sigma : P \rightarrow M$ and $\hat{\varphi} : M \rightarrow K$ such that $\varphi = \hat{\varphi}\sigma$. Let L' be a supplement of $\text{Ker } \hat{\varphi}$ in M and let $L = \sigma^{-1}(L')$. For an arbitrary $p \in P$, $\sigma(p)$ may be written as $\sigma(p) = l' + k'$, with $l' \in L'$ and $k' \in \text{Ker } \hat{\varphi}$. It follows then that $\varphi(p) = \hat{\varphi}\sigma(p) = \hat{\varphi}(l') + \hat{\varphi}(k') = \hat{\varphi}(l')$. Choose $p_1 \in \sigma^{-1}(l') \subset L$. Then $\sigma(p_1) = l'$. On the other hand, $\varphi(p_1) = \hat{\varphi}\sigma(p_1) = \hat{\varphi}(l') = \varphi(p)$. So $p - p_1 \in \text{Ker } \varphi$ and so $L + \text{Ker } \varphi = P$. The fact that $P/L \cong M/L'$ follows since L is the kernel of the onto map $\pi_L \cdot \sigma : P \rightarrow M/L'$. Conversely, let us assume that for every submodule $K \subset N$ and for every epimorphism $\varphi : P \rightarrow K$ there exist an epimorphism $\hat{\varphi} : M \rightarrow K$ such that for every supplement L' of $\text{Ker } \hat{\varphi}$ in M there exists a submodule $L \subset P$ such that $P/L \cong M/L'$ and $L + \text{Ker } \varphi = P$. Let $\varphi : P \rightarrow K$ be an epimorphism and $\hat{\varphi} : M \rightarrow K$ be the corresponding epimorphism. All we need is to produce another epimorphism $\sigma : P \rightarrow M$ such that $\varphi = \hat{\varphi}\sigma$. Let L' be a supplement for $\text{Ker } \hat{\varphi}$ and let L be the corresponding submodule of P . Let $\theta : P/L \rightarrow M/L'$ be an isomorphism. The Chinese remainder theorem yields that the map $m + \text{Ker } \hat{\varphi} \cap L' \mapsto (M + \text{Ker } \hat{\varphi}, m + L')$ is an isomorphism between $M/(\text{Ker } \hat{\varphi} \cap L')$ and $M/\text{Ker } \hat{\varphi} \times M/L'$. Also, $M/\text{Ker } \hat{\varphi} \cong K$ via $m + \text{Ker } \hat{\varphi} \mapsto \hat{\varphi}(m)$. So, one gets an isomorphism $\beta : M/(\text{Ker } \hat{\varphi} \cap L') \rightarrow K \times M/L'$ such that $\beta(m + \text{Ker } \hat{\varphi} \cap L') = (\hat{\varphi}(m), \pi_{L'}(m))$. The isomorphism θ induces an onto map $\Psi = \theta\pi_L : P \rightarrow M/L'$. Since $\text{Ker } \varphi + L = P$, the map $\alpha : P \rightarrow K \times M/L'$ given by $\alpha(p) = (\varphi(p), \Psi(p))$ is onto. The induced epimorphism $\alpha' = \beta^{-1}\alpha : P \rightarrow M/(\text{Ker } \hat{\varphi} \cap L')$ may then be lifted to a map $\sigma : P \rightarrow M$. Since $\text{Ker } \hat{\varphi} \cap L' \ll M$ σ is indeed an epimorphism. It only remains to show that $\hat{\varphi}\sigma = \varphi$. Let us refer for the rest of this proof to $\pi_{\text{Ker } \hat{\varphi} \cap L'}$ simply as π . We do know that $\pi\sigma = \sigma' = \beta'\alpha$

hence $\beta\pi\sigma = \alpha$. Let $p \in P$ be arbitrary. Then $\beta(\sigma(p) + \text{Ker } \hat{\varphi} \cap L') = \alpha(p) = (\varphi(p), \Psi(p))$. On the other hand, $\beta(\sigma(p) + \text{Ker } \hat{\varphi} \cap L') = (\hat{\varphi}(\sigma(p), \sigma(p) + L')$. Comparing the first component in both expressions yields the desired equality. Thus, M is weakly N -projective. \square

2.6 Corollary. Let M be a hollow module with projective cover P and N be an arbitrary module. Then M is weakly N -projective if and only if any submodule K of M , which is a homomorphic image of P , is a homomorphic image of M .

Proof. Straightforward from the above proposition. \square

Modules which are weakly projective relative to a fixed module are closed under finite direct sums and under superfluous covers but not under direct summands. So, in particular, finite direct sums of weakly projective modules are weakly projective and superfluous covers of weakly projective modules are weakly projective.

2.7 Proposition

- (1) Let $M_i, i = 1, 2, \dots, n$ be a family of weakly N projective modules. Then the direct sum $\bigoplus_{i=1}^n M_i$ is weakly N -projective.
- (2) Let M/N be a weakly K -projective module where $N \ll M$. Then M is weakly K -projective.
- (3) It is possible to have a direct sum $M \oplus K$ being weakly N -projective while M is not weakly N -projective.
- (4) If a module is weakly projective relative to its own projective cover, then the module is indeed projective.

Proof. (1) and (2) are straightforward. To prove (3), let $R = \mathbf{Z}/(4)$. Then $\mathbf{Z}/(2) \times \mathbf{Z}/(4)$ is weakly R -projective (see Proposition 2.11, below) but $\mathbf{Z}/(2)$ is not weakly R -projective, in light of (4). In order to prove (4), consider a module M with projective cover $\pi : P \rightarrow M$. If we assume that M is weakly P -projective, then the identity map on P factors through M and this yields that $M \cong P$. \square

2.8 Remark. Over a right perfect ring, any arbitrary direct sum of weakly N -projective modules is weakly N -projective.

Proof. This can be proven in the same way as the proof of Proposition 2.7(1) since, over a right perfect ring, the projective cover of a direct sum of modules is the direct sum of the projective covers of the individual modules. \square

A finitely generated direct summand S of the projective cover of a weakly projective module M yields a direct summand (isomorphic to S) of M .

2.9 Lemma. Let M be a weakly projective module whose projective cover $P(M) = S \oplus K$, where S is finitely generated. Then M has a direct summand isomorphic to S .

Proof. Since S is finitely generated, M is weakly S -projective (Proposition 2.3). Thus the projection map $p : P(M) \rightarrow S$ factors through M , yielding an epimorphism $\hat{p} : M \rightarrow S$. Since S is projective we get that $M \cong S \times \text{Ker } \hat{p}$, proving our claim. \square

The next result points out the fact that weakly projective but not projective modules are 'large'.

2.10 Proposition. Every finitely generated projective module is indeed projective. Over a semiperfect ring R , a finite Goldie dimensional weakly projective module is indeed projective.

Proof. If M is finitely generated, then $P(M)$ is also finitely generated and so, by Proposition 2.7(4), M is projective. Suppose $R = \bigoplus_{i=1}^n e_i R$ is the representation of the semiperfect ring R as a direct sum of indecomposable projective modules. Let N be a finite Goldie dimensional weakly projective R -module. Write $P(N) = \bigoplus_{i=1}^n (e_i R)^{\alpha_i}$ as a direct sum of indecomposable projective modules. If any of the α_i 's were infinite, by Lemma 2.9, N would contain sums of arbitrarily many submodules, contradicting that N has finite Goldie dimension. Therefore, $P(N)$ is finitely generated and hence N is projective. \square

2.11 Proposition. Let R be a right self-injective local ring, but not a division ring, with nonzero right socle. Then for every integer $n > 0$, $R/S \times R^n$ is weakly R^n -projective but not weakly R^{n+1} -projective.

Proof. Since R is local and right self-injective, R is uniform, and so the right socle S is a simple right ideal. However, it is easy to see that S is also a simple left ideal. For, if $s \in S$ then $r \cdot \text{ann}(s) \supseteq J$, since $SJ = 0$. But J , being maximal, $r \cdot \text{ann}(s) = J$. Then for any $0 \neq x \in S$, the mapping $sR \rightarrow xR$ given by $sr \rightarrow xr$ is well defined and so by right self-injectivity of R there exists $x' \in R$ such that $x's = x$. This implies $x \in Rs$ and, therefore, $S = Rs$, because S is two-sided. This yields S is also a simple left ideal. Thus $JS = 0$.

Since R is not a division ring, the simple right ideal S is a proper right ideal of R and so is contained in J . This implies that the natural epimorphism $R \rightarrow R/S$ gives a projective cover. Furthermore, for every n the canonical epimorphism $R^{n+1} \rightarrow R/S \times R^n$ defines a projective cover. We shall prove the result by induction on n . Consider first the case where $n = 1$. Let $\varphi : R \times R \rightarrow R$ be a homomorphism. We put $a = \varphi(1, 0)$ and $b = \varphi(0, 1)$, and we define $u = (1, 0)$ or $(-a^{-1}b, 1)$ according as $a \in J$ or $a \notin J$. Then $\varphi(u) = a$ if $a \in J$ and $\varphi(u) = \varphi(-(1, 0)a^{-1}b + (0, 1)) = -aa^{-1}b + b = 0$ if $a \notin J$. Now clearly the submodule uR of $R \times R$ is isomorphic to R and the submodule uS of uR is isomorphic to S . Moreover, we know that $uS \subset \text{Ker } \varphi$, because if $a \in J$ the $\varphi(uS) = \varphi(u)S = aS \subset JS = 0$, while if $a \notin J$ then $\varphi(uS) = \varphi(u)S = 0S = 0$. Since $uR \cong R$ is injective, $R \times R = uR \oplus K$ for some $K \subset R \times R$, which is necessaril

$\cong R$. It follows that $(R \times R)/uS = (uR \oplus K)/uS = uR/uS \times K = R/S \times R$. Thus $R/S \times R$ is weakly R -projective by Theorem 2.2.

Let us assume next that $n > 1$ and our proposition holds for $n - 1$. Consider a homomorphism $\varphi : R^{n+1} \rightarrow R^n$. Let $\pi_i : R^n \rightarrow R$ be the projection onto the i -th component. We need to consider two cases depending on whether there is a value of i ($1 \leq i \leq n$) for which $\pi_i \varphi$ is onto. In the affirmative case, assume without loss of generality that π_1 is onto. Let us refer to the projection onto the complement simply as $\pi : R \times R^{n-1} \rightarrow R^{n-1}$. Let $\alpha : R^n \rightarrow \text{Ker } \pi_1 \varphi$ be an isomorphism. Using our inductive hypothesis on $\pi \varphi \alpha : R^n \rightarrow R^{n-1}$, we obtain a $Y \subset \text{Ker } \pi \varphi \alpha$ such that $R^n/Y \cong R/S \times R^{n-1}$. Let $X = \alpha(Y)$. Then clearly $X \subset \text{Ker } \pi \varphi$ and also $X \subset \alpha(R^n) = \text{Ker } \pi_1 \varphi$. But $\text{Ker } \pi \varphi \cap \text{Ker } \pi_1 \varphi = \text{Ker } \varphi$, and thus we have $X \subset \text{Ker } \varphi$. Since R^n whence $\text{Ker } \pi_1 \varphi$ is injective, $R^{n+1} = \text{Ker } \pi_1 \varphi \oplus L$ for some submodule $L \subset R^{n+1}$ and necessarily $L \cong R$. Then we have $R^{n+1}/X \cong \text{Ker } \pi_1 \varphi / X \times L \cong R^n/Y \times R \cong R/S \times R^{n-1} \times R \cong R/S \times R^n$, proving our claim again by Theorem 2.2. To conclude, consider when none of the natural projections $\pi_i : R^n \rightarrow R$ satisfies that $\pi_i \varphi$ is onto, or what is the same, $\pi_i(\varphi(R^{n+1})) \subset J$ for $i = 1, 2, \dots, n$. Denote by S^{n+1} the external product $S \times S \times \dots \times S$ of $n+1$ copies of S . Then we have $\pi_i(\varphi(S^{n+1})) = \pi_i(\varphi(R^{n+1}S)) = \pi_i(\varphi(R^{n+1}))S \subset JS = 0$ for every i , which means that $\varphi(S^{n+1}) = 0$. Let $X = S \times 0 \times \dots \times 0$. Then $X \subset S^{n+1} \subset \text{Ker } \varphi$ and $R^{n+1}/X \cong R/S \times R^n$. This shows that $R/S \times R^{n+1}$ is weakly R^n -projective, as claimed.

Suppose that $R/S \times R^n$ is weakly R^{n+1} -projective. Then $R/S \times R^n$ and hence R/S is projective by Proposition 2.7(4). But this implies that $S = 0$, which is a contradiction. This completes the proof. \square

An important fact in the theory of weakly injective modules is that a quasi-injective weakly injective module is indeed injective [6]. The dual result is

2.12 Proposition. Let N be a module. Then any quasi-projective weakly N -projective module is indeed N -projective.

Proof. Let M be a quasi-projective module M with projective cover $\pi : P \rightarrow M$ and assume that M is weakly N -projective. Consider a map $\varphi : M \rightarrow N/K$, for some submodule $K \subset N$. The projectivity of P guarantees the existence of a map $\hat{\varphi} : P \rightarrow N$ such that $\varphi \pi = \pi_K \hat{\varphi}$. Now, since M is weakly N -projective there exist an epimorphism $\sigma : P \rightarrow M$ and a map $\hat{\psi} : M \rightarrow N$ such that $\hat{\psi} \sigma = \hat{\varphi}$. Since M is quasi-projective there exists $\sigma' : M \rightarrow M$ such that $\sigma' \pi = \sigma$, (Theorem 2.0). One easily checks that the map $\hat{\psi} \sigma' : M \rightarrow N$ lifts φ , proving our claim. \square

3. Every Module is a Direct Summand of a Weakly Projective Module

It is shown in [9] that over arbitrary rings every semisimple module is a direct summand of a weakly injective module and every module (without restriction) is a direct summand of a tight module. This implies that if the ring is right q.f.d.

(i.e., if all cyclic right modules have finite uniform dimension) then every module is a summand of a weakly injective one. For right q.f.d. rings, one can actually get a stronger result 'with the quantifiers commuted'. Namely, one can show that not only is it true that for every right module M there exists a right module K such that $M \oplus K$ is weakly injective, but actually that a fixed module K may be chosen such that, for every right module M , the sum $M \oplus K$ is weakly injective [7]. However, the construction of an ad hoc K for a given M , as in [9], has an interest of its own right and can be used in applications where the second construction would not apply. In the dual case, we will show that for right perfect rings, it is also true that every right module is a summand of a weakly projective module.

3.1 Theorem. Over a right perfect ring R , there exists a module K such that the direct sum of K plus any other module yields a weakly projective module.

Proof. Since R is right perfect, we may write $R = \bigoplus_{i=1}^{\aleph} (e_i R)^{\alpha_i}$, where $\{e_1 R, \dots, e_{\aleph} R\}$ is a complete set of representatives of indecomposable projective right R -modules. Let $L = \bigoplus I$, where $I \subset R^n$ for all $n \in \mathbb{Z}^+$ be the external sum of all submodules of finitely generated free right R -modules. Let \aleph be an infinite cardinal such that $\aleph > |R|$. Define $K = L \oplus [P(L)]^{(\aleph)}$, where $P(L)$ is the projective cover of L . Consider an arbitrary right R -module M and an integer $n \in \mathbb{Z}^+$. Our aim is to show that the direct sum $N = M \oplus K$ is weakly R^n -projective. Consider an epimorphism $\varphi : P(N) \rightarrow I$, where $I \subset R^n$. Let $\pi : P(I) \rightarrow I$ be the projective cover map. The projectivity of $P(N)$ yields a map $\hat{\varphi} : P(N) \rightarrow P(I)$ such that $\pi \hat{\varphi} = \varphi$. Furthermore, since $\text{Ker } \pi \ll P(I)$, one gets that $\hat{\varphi}$ is an epimorphism. Since $P(I)$ is projective, $\hat{\varphi}$ splits and, therefore, we may write $P(N) = P \oplus \text{Ker } \hat{\varphi}$, for some submodule $P \subset P(N)$ isomorphic to $P(I)$. Over a semiperfect ring all projective modules are decomposable as direct sums of indecomposable projective ones. So let us write $P(I) \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\alpha_i)}$ and $\text{Ker } \hat{\varphi} \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\beta_i)}$. Suppose further that $P(L) \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\gamma_i)}$. Then $P(K) \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\gamma_i, \aleph)} \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\delta_i)}$, where $\delta_i \geq \aleph$. Let $P(M) \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\epsilon_i)}$.

Since there exists an epimorphism $\Psi : R^{(|I|)} \rightarrow I$, $P(I)$ is isomorphic to a summand of $R^{(|I|)} \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\alpha_i, |I|)}$. Therefore, $\alpha_i \leq n$, $|I| \leq n$, $|R^n| = n \cdot n |R| < \aleph$ for each i . The decompositions $P(N) = P(M) \oplus P(K)$ and $P(N) = P \oplus \text{Ker } \hat{\varphi}$ imply that $\bigoplus_{i=1}^{\aleph} (e_i R)^{(\alpha_i, \cup \beta_i)} \cong \bigoplus_{i=1}^{\aleph} (e_i R)^{(\delta_i, \cup \epsilon_i)}$. Since each $\alpha_i < \aleph$, while $|D_i \cup F_i| \geq \aleph$, we must conclude that $|D_i \cup F_i| = \beta_i$. So, $\text{Ker } \hat{\varphi} \cong P(N)$ and one can think of $\hat{\varphi}$ as the projection $p : P(N) \times P(I) \rightarrow P(I)$. It then follows that $\text{Ker } \varphi \cong P(N) \times \text{Ker } \pi$. Now, N is a homomorphic image of $P(N)$ and, by definition of K , there exists a submodule $N' \subset N$ such that $I \oplus N' = N$. So, there exists a submodule $K' \subset P(N)$ such that $P(N)/K' \cong N'$.

Let $X = K' \times \text{Ker } \pi \subset \text{Ker } \varphi$. Then $P(N)/X = [P(N) \times P(I)]/[K' \times \text{Ker } \pi] \cong N' \times I \cong N$, as desired. □

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