Can. J. Math. Vol. 46 (5), 1994 pp. 971–981

WEAKLY PROJECTIVE AND WEAKLY INJECTIVE MODULES

S. K. JAIN, S. R. LÓPEZ-PERMOUTH, K. OSHIRO AND M. A. SALEH

ABSTRACT. A module M is said to be *weakly N-projective* if it has a projective cover $\pi: P(M) \twoheadrightarrow M$ and for each homomorphism $\varphi: P(M) \longrightarrow N$ there exists an epimorphism $\sigma: P(M) \twoheadrightarrow M$ such that $\varphi(\ker\sigma) = 0$, equivalently there exists a homomorphism $\hat{\varphi}: M \longrightarrow N$ such that $\hat{\varphi}\sigma = \varphi$. A module M is said to be *weakly projective* if it is weakly N-projective for all finitely generated modules N. Weakly N-injective and weakly injective modules are defined dually. In this paper we study rings over which every weakly injective right R-module is weakly projective. We also study those rings over which every weakly projective right module is weakly injective. Among other results, we show that for a ring R the following conditions are equivalent:

(1) R is a left perfect and every weakly projective right R-module is weakly injective.

(2) R is a direct sum of matrix rings over local QF-rings.

(3) R is a QF-ring such that for any indecomposable projective right module eR and for any right ideal I, $\mathrm{soc}(eR/eI) = (eR/eJ)^n$ for some positive integer n.

(4) R is right artinian ring and every weakly injective right R-module is weakly projective.

(5) Every weakly projective right R-module is weakly injective and every weakly injective right R-module is weakly projective.

0. **Introduction.** The purpose of this paper is to study certain relations between the concepts of weakly injective and weakly projective modules. Let M and N be right R-modules. We say that M is *weakly N-projective* if M has a projective cover $\pi: P(M) \twoheadrightarrow M$ and every homomorphism $\varphi: P(M) \longrightarrow N$ can be factored through M via some epimorphism (not necessarily equal to π). Equivalently, a module M is weakly N-projective if it has a projective cover $\pi: P(M) \twoheadrightarrow M$ and given any homomorphism $\varphi: P(M) \longrightarrow N$, there exists $X \subseteq \ker\varphi$ such that $P(M)/X \cong M$. A module M is said to be *weakly projective* if it is weakly N-projective for every finitely generated module N. Dually, a module M is said to be *weakly N-injective* if for each homomorphism $\varphi: N \longrightarrow E(M)$ there exists a monomorphism $\sigma: M \longrightarrow E(M)$ and a homomorphism $\hat{\varphi}: N \longrightarrow M$ such that $\varphi = \sigma\hat{\varphi}$. M is called *weakly injective* if it is weakly N-injective for each finitely generated module N. Equivalently, a module M is weakly injective if for every finitely generated submodule N of the injective hull $E(M)$ of M there exists $X \subseteq E(M)$ such that $N \subseteq X \cong M$. A module M is said to be *tight* if for all finitely generated modules N, N embeddable in $E(M)$

The first author was partially supported by NSF grant No. INT 9210491.

The second author author was partially supported by the Ministry of Education and Science of Spain and the Centre de Recerca Matemática, Barcelona, Spain.

The third author was supported by the Ministry of Education, Science and Culture of Japan.

Received by the editors March 1, 1993.

AMS subject classification: 16D40, 16D50, 16L30, 16L60, 16P20.

© Canadian Mathematical Society 1994.

implies N is embeddable in M. Clearly, every weakly injective module is tight. However, the converse does not hold (*cf.*, [11]).

It is well-known that quasi-Frobenius rings can be characterized by any one of the following properties: (1) each projective module is injective; (2) each injective module is projective. Theorem 3.7 of our paper characterizes those left perfect rings R over which each weakly projective right R-module is weakly injective precisely as direct sums of matrix rings over local QF-rings. Under the hypothesis that each weakly injective right module over a ring R is weakly projective, it is shown that R is a two-sided perfect right pseudo-Frobenius ring (PF-ring). (Proposition 4.2 and Theorem 4.5). Theorem 5.4 shows that R is a finite direct sum of matrix rings over local QF-rings if and only if each weakly projective right module is weakly injective and also each weakly injective right module is weakly projective. The proofs of our main results depend upon several key lemmas, and other known results contained in [1, 6, 7, 8, 9, 11].

1. **Definitions and notation.** A submodule N of a module M is said to be a *small* (superfluous) submodule if the only submodule K of M such that $K + N = M$ is $K = M$. A small module N in M is denoted by $N \ll M$. A *superfluous cover* of a module M is a module P together with an epimorphism $p: P \longrightarrow M$ such that $\ker p$ is small in P. Equivalently, one may think of a superfluous cover for M as being a module P such that $P/K \cong M$ for some small submodule $K \subsetneq P$. A projective superfluous cover of M will be referred to, as is customary, as a projective cover, denoted by $P(M)$. An essential (or large) submodule N of M is denoted by $N \subseteq' M$. $E(M)$ will denote the injective hull of M. $\mathrm{soc}(M)$ and $\mathrm{rad}(M)$ will denote respectively the socle and radical of a module M. A ring R is called a *right q.f.d. ring* if and only if every cyclic right R-module has a finitely generated (possibly zero) socle. This is equivalent to saying that every cyclic (finitely generated) right module has finite Goldie dimension. All rings with right Krull dimension are right q. f. d. In particular, right noetherian rings are right q. f. d. A ring R is called a *right CEP-ring* if every cyclic right R-module is essentially embeddable in a projective module. For any submodule K of a module N the natural inclusion map will be denoted by $i_K: K \longrightarrow N$ and the natural projection by $\pi_K: N \longrightarrow N/K$. The arrow $\twoheadrightarrow$ serves to emphasize that the corresponding map is an epimorphism.

In this paper we assume all modules are unital right R-module unless otherwise indicated. Our terminology is same as [3], [5] and [10], unless otherwise defined.

2. **Preliminaries.** In this section we begin with listing some of the basic results on weakly injective and weakly projective modules which we shall need throughout this paper.

LEMMA 2.1 [7, PROPOSITION 1.1]. *Let M and N be R-modules. If M is weakly N-injective, then M is weakly N/K-injective for each submodule K of N. In particular, M is weakly injective if and only if M is weakly R^n-injective for all $n \in Z^+$.*

LEMMA 2.2. *(1) The finite direct sum of weakly injective modules is weakly injective.*

(2) Let $N \subseteq' M$ be right modules such that N is weakly injective. Then M is weakly injective.

(3) A right module M is weakly injective iff for any $n \in Z^+$ and $x_1, \ldots, x_n \in E(M)$ there exists a submodule $X \subseteq E(M)$ such that $x_i \in X \cong M$, for all i. In particular, M is weakly R-injective if and only if every cyclic submodule of $E(M)$ is contained in a submodule X of $E(M)$ isomorphic to M.

PROOF. (1) See [8, Proposition 1.7 and Remark 1.5].

(2) Clear.

(3) See [8, Remark 1.5].

LEMMA 2.3 [8, THEOREM 1.11]). *A semiperfect ring R is right* CEP *if and only if R is right artinian and every indecomposable projective right module is weakly R-injective.*

LEMMA 2.4 [11, THEOREM 3.1]. *Over a right q.f.d. ring R, a module M is weakly injective if and only if each finitely generated module N which is embeddable in E(M) is indeed embeddable in M (i.e. M is tight).*

LEMMA 2.5 [1, THEOREM 1]. *The following statements are equivalent for a ring R:*

(1) R is a right q.f.d. ring.

(2) Each direct sum of indecomposable injective right R-modules is weakly injective.

(3) Each direct sum of weakly injective R-modules is weakly injective.

LEMMA 2.6 [11, PROPOSITIONS 2.1 AND 2.3]. *Let R be a ring and let M be a semisimple right module. Then there exists an infinite cardinal $\aleph$ such that $M \oplus E(M^{(\aleph)})$ is weakly injective. Consequently, this result holds also for any right R-module with essential socle. Moreover, if R is a right q.f.d. ring then the result holds for any right module M.*

The remark which follows gives that over a q. f. d. ring there exists a weakly injective module K such that $M \oplus K$ is weakly injective for all modules M.

REMARK 2.7. Let R be a q. f. d. ring and $\mathcal{F}$ be the family of all indecomposable injective R-module (up to isomorphism). If $K = \oplus_{E \in \mathcal{F}} \Sigma(E)^{(\omega)}$ where ω is countably infinite then $M \oplus K$ is weakly injective for all modules M.

PROOF. Since every indecomposable injective module is the injective hull of a cyclic module, the replacement axiom of set-theory yields that $\mathcal{F}$ is a set (rather than a class). Over a q. f. d. ring each finitely generated module N is embeddable in a finite direct sum of indecomposable injective modules. This yields that N is embeddable in K. Thus $M \oplus K$ is weakly injective.

LEMMA 2.8. [2, THEOREM 3.3]. *Let R be a left or right perfect ring. Then R is right weakly injective as a right R-module iff R is right self injective.*

For reader's convenience we provide here the proofs of basic results on weakly projective modules contained in [9], which will be used in the sequel.

LEMMA 2.9. *A right module M is weakly projective if and only if M is weakly R^n-projective for all positive integers n.*

PROOF. We need only show that if M is weakly R^n-projective for all positive integers n, then it is weakly projective. Let N be a finitely generated module and let $\varphi: P(M) \longrightarrow N$. Since N is finitely generated, there exists an epimorphism $\rho: R^n \twoheadrightarrow N$ for some $n \in Z^+$. The projectivity of $P(M)$ yields the existence of a homomorphism $\varphi': P(M) \longrightarrow R^n$ such that $\rho\varphi' = \varphi$. Since M is weakly projective, there exists $X \subseteq \ker \varphi'$ such that $P(M)/X \cong M$. On the other hand $\ker \varphi' \subseteq \ker \varphi$, proving M is weakly N-projective.

LEMMA 2.10 [9, PROPOSITION 2.7 AND LEMMA 2.9]. *(1) Over an arbitrary ring, a finite direct sum of weakly projective right modules is weakly projective. Over a right perfect ring, any direct sum of weakly projective right modules is weakly projective.*

(2) Let M be a weakly projective right module over a semiperfect ring. If the projective cover $P(M)$ may be expressed as $S \oplus K$, with S finitely generated, then M has a direct summand isomorphic to S. In particular, a finitely generated or an indecomposable weakly projective right module is projective.

Proof. (1) This is straightforward.

(2) Since S is finitely generated, M is weakly S-projective (Lemma 2.8). Thus the projection map $\pi_1: P(M) \longrightarrow S$ factors through M, yielding an epimorphism $\pi_1': M \longrightarrow S$. Since S is projective we get that $M \cong S \oplus \operatorname{Ker} \pi'$, proving our claim.

REMARK 2.11. Let R and S be Morita equivalent rings with $\mathbf{T}: \text{Mod-}R \longrightarrow \text{Mod-}S$, being an equivalence between the categories of right R-modules and right S-modules. If M is a weakly projective (weakly injective) right R-module then $\mathbf{T}(M)$ is a weakly projective (weakly injective) S-module.

PROOF. Straightforward from Lemmas 2.1 and 2.9.

LEMMA 2.12 [6, COROLLORY 1.12]. *A left perfect and right self-injective ring R is* QF *iff every cyclic right R-module embeds in a free module.*

LEMMA 2.13 [12, THEOREM 2(4)]. *Every right module over a* QF*-ring can be expressed as a direct sum of a projective module and a singular module.*

We conclude this section with some examples of weakly projective modules which are not projective.

EXAMPLES 2.14. (1) Let R be a uniserial ring which is not a division ring (*e.g.* $Z/(p^n), p$ prime), and $S = \operatorname{soc}(R)$. Then, as a right R-module, $R/S \times R$ is weakly R-projective but not R-projective (see [9, Proposition 2.11]).

Furthermore, the right R-module $M = R/S \times R^{(\omega)}$, where ω is infinite cardinal, is weakly projective but not projective.

(2) Let R be a right perfect ring, and L be the direct sum of submodules (up to isomorphism) of all finitely generated free right R-modules. Then $M = L \oplus \left(P(L)\right)^{(\alpha)}$ where

$\alpha > |R|$ is weakly projective. Indeed for each right R-module K, $M \oplus K$ is also weakly projective. (see [9, Theorem 3.1]).

We should emphasize that the definition of weakly projective module requires that the module has a projective cover.

3. **When weakly projective modules are weakly injective.** It is an open question whether a (one-sided) perfect, one-sided self injective ring is QF (*cf.*, [4], [6]). The next theorem is a result in that direction.

THEOREM 3.1. *Let R be a left perfect ring such that every projective right R-module is weakly R-injective. Then R is a* QF*-ring.*

Proof. We first show that every cyclic right R-module embeds in a free module. Since R is projective then, by hypothesis, R is right weakly injective. Using Lemma 2.8 we conclude that R is right self-injective. Then R is a right PF-ring [10, Theorem 12.5.2]. If C is a cyclic right R-module, then $S = \operatorname{soc}(C)$ is essential in C and S embeds in a free right R-module F, say. Now $E(C) = E(S) \subset E(F)$ and F is weakly R-injective. Thus $C \subset X \cong F$ for some submodule X. By Lemma 2.12, we conclude that R is a QF-ring.

PROPOSITION 3.2. *Let R be a* QF*-ring and let M_R be an R-module. Express* $M = E \oplus K$, *where* $E = \oplus \sum_{i=1}^{k} (e_i R)^{(\alpha_i)}$ *is a projective module and K is a singular module. If we write* $K/\operatorname{rad} K = \oplus \sum_{i=1}^{k} (\frac{e_i R}{e_i J})^{(\beta_i)}$ *and* $\operatorname{soc}(K) = \oplus \sum_{i=1}^{k} (\frac{e_i R}{e_i J})^{(\gamma_i)}$, *then*

(a) *M is weakly projective if and only if for all* $i = 1, \ldots, k$ *if* $\beta_i \neq 0$ *then* α_i *is infinite.*

(b) *M is weakly injective if and only if for all* $i = 1, \ldots, k$ *if* $\gamma_i \neq 0$ *then* α_i *is infinite.*

PROOF. (a) The necessity is clear. For, if α_j is finite and $\beta_j \neq 0$ then $P(M) = \left(\oplus \sum_{i=1}^{k} (e_i R)^{(\alpha_i)}\right) \oplus \left(\oplus \sum_{i=1}^{k} (e_i R)^{(\beta_i)}\right)$ yields by Lemma 2.10 (2), $e_j R^{(\alpha_j+1)} \subseteq^{\oplus} M$, a contradiction.

For the converse, let us start by writing $M = \oplus \sum_{i \in A} (e_i R)^{(\alpha_i)} \oplus \sum_{j \in B} (e_j R)^{(\alpha_j)} \oplus K$, where $A = \{i \mid \beta_i \neq 0\}$ and $B = \{i \mid \beta_i = 0\}$. It suffices to show $\oplus \sum_{i \in A} (e_i R)^{(\alpha_i)} \oplus K$ is weakly projective. So we may assume $\beta_i \neq 0$ for all i. Consider an epimorphism $\varphi : P(M) \twoheadrightarrow I$, where $I \subseteq R^n$. Let $\pi : P(I) \twoheadrightarrow I$ be a projective cover map. The projectivity of $P(M)$ yields a map $\hat{\varphi} : P(M) \longrightarrow P(I)$ such that $\pi \hat{\varphi} = \varphi$. Since $\operatorname{Ker} \pi \ll P(I)$, $\hat{\varphi}$ is an epimorphism. Furthermore because $P(I)$ is projective, $\hat{\varphi}$ splits and therefore we may write $P(M) = P \oplus \operatorname{Ker} \hat{\varphi}$ for some submodule $P \subseteq P(M)$ isomorphic to $P(I)$. Let us write $P(I) = \oplus \sum_{i=1}^{k} (e_i R)^{(n_i)} \cong P$, and $\operatorname{Ker} \hat{\varphi} \cong \oplus \sum_{i=1}^{k} (e_i R)^{(\theta_i)}$, and $P(M) \cong \oplus \sum_{i=1}^{k} (e_i R)^{(\delta_i)}$, where δ_i is infinite. It follows that $\operatorname{Ker} \hat{\varphi} \cong P(M)$. Thus there exists $X \subseteq \operatorname{Ker} \hat{\varphi}$ such that $\operatorname{Ker} \hat{\varphi}/X \cong M$. Now, $X \subseteq \operatorname{Ker} \hat{\varphi} \subseteq \operatorname{Ker} \varphi$, and $P(M)/X = \operatorname{Ker} \hat{\varphi} \oplus P/X \oplus 0 \cong (\operatorname{Ker} \hat{\varphi}/X) \oplus P \cong M \oplus P \cong M$, proving our claim.

(b) To prove necessity, assume on the contrary that there exists γ_i such that $\gamma_i \neq 0$ and α_i is finite. Put $N = (e_i R)^{(\alpha_i+1)}$. By the weak injectivity of M, $(e_i R)^{(\alpha_i+1)}$ embeds in M, contradicting our assumption.

To prove the converse, one can argue in a similar way as in part (a) and assume that, for all i, $\lambda_i \neq 0$. Consider a finitely generated submodule $N \subseteq E(M) = \oplus \sum_{i=1}^{k} (e_i R)^{(\lambda_i)}$, where each λ_i is infinite. Then there exists positive integers $n_1, \ldots, n_k$ such that $N \subsetneq \oplus \sum_{i=1}^{k} (e_i R)^{(n_i)}$ and thus we conclude that M is tight, hence weakly injective by Lemma 2.4.

Recall a ring R is called *local* if it has a unique maximal right ideal.

THEOREM 3.3. *Let R be a local* QF*-ring and let M be a right R-module. Then M is weakly projective if and only if M is weakly injective.*

PROOF. By Lemma 2.12, we may express $M = E \oplus K$, where K is a singular module and E is a free module. By Proposition 3.2, M is weakly projective if and only if $K = 0$ or $E = R^{(\alpha)}$ with α infinite. This is equivalent to M being weakly injective.

COROLLARY 3.4. *Let R be a direct sum of matrix rings over local* QF*-rings and let M be a right R-module. Then M is weakly projective if and only if M is weakly injective.*

PROOF. The proof follows from Morita equivalence between R and $M_n(R)$ (see Remark. 2.11).

PROPOSITION 3.5. *Let R be a left perfect ring such that every weakly projective right R-module is weakly R-injective. Then for indecomposable projective modules eR and fR, if fR/fJ embeds in eR/eI for some right ideal I, then $eR \cong fR$. Equivalently,* $\mathrm{soc}(eR/eI) \cong (eR/eJ)^n$ *for some positive integer n.*

PROOF. Note that, by Theorem 3.1, R is QF. Suppose on the contrary that $\mathrm{soc}(fR)$ embeds in eR/eI and fR is not isomorphic to eR. We will show that $N = (eR)^{(\omega)} \oplus eR/eI$ is weakly projective but not weakly injective. Since $E(N) = (eR)^{(\omega)} \oplus fR \oplus K, fR \subsetneq E(N)$. On the other hand, fR is not embeddable in N, and so M is not weakly injective.

To show that N is weakly projective, consider an epimorphism $\varphi: P(N) \twoheadrightarrow K$, where $K \subsetneq R^n$. Let $\pi: P(K) \twoheadrightarrow K$ be the projective cover map. The projectivity of $P(N)$ yields a map $\hat{\varphi}: P(N) \longrightarrow P(K)$ such that $\pi\hat{\varphi} = \varphi$. Since $\operatorname{Ker}\pi \ll P(K)$, we get that $\hat{\varphi}$ is an epimorphism and therefore we may write $P(N) = P \oplus \operatorname{Ker}\hat{\varphi}$ for some submodule $P \subsetneq P(N)$ isomorphic to $P(K)$. Therefore, $(eR)^{(\omega)} = \operatorname{Ker}\hat{\varphi} \oplus P$, where P is finitely generated. This implies that $(eR)^{(\omega)} \cong \operatorname{Ker}\hat{\varphi}$. Let $X = 0 \oplus eI$. Then $P(N)/X = (eR)^{(\omega)} eR)/(0 \oplus eI) \cong (eR)^{(\omega)} \oplus (eR/eI) = N$, as desired. Hence N is weakly projective which is not weakly injective, a contradiction to our hypothesis. Therefore, $\mathrm{soc}(eR)$ embeds in eR/eJ and thus, $\mathrm{soc}(eR) \cong eR/eJ$. Consequently, $\mathrm{soc}(eR/eI) \cong (eR/eJ)^n$ for some positive integer n.

PROPOSITION 3.6. *Let R be a* QF*-ring such that for any indecomposable projective right module eR and for any right ideal I,* $\mathrm{soc}(eR/eI) = (eR/eJ)^n$ *for some positive integer n. Then R is a direct sum of matrix rings over local* QF*-rings.*

PROOF. Write $R = \oplus \sum_{i=1}^{n} e_i R$, where $\{e_i \mid i = 1, \ldots, n\}$ is a complete set of orthogonal primitive idempotents, and let $A = \{e_i R \mid i = 1, \ldots, k\}$ be a complete set of representatives for the indecomposable projective right R-modules. Let $[e_i R] = \sum e_j R$, where the summation runs over all j for which $e_j R \cong e_i R$. Renumbering if necessary we may write $R = [e_1R] \oplus \cdots \oplus [e_kR]$ where $k \leq n$. By our hypothesis (see also Proposition 3.5), $[e_iR]$ is an ideal in R and so $R \cong M_{n_1}(e_1Re_1) \oplus \cdots \oplus M_{n_k}(e_kRe_k)$ where n_i is the number of summands in $[e_iR]$.

THEOREM 3.7. *Let R be a left perfect ring such that every weakly projective right module is weakly injective. Then R is a direct sum of matrix rings over local* QF*-rings.*

PROOF. The proof follows directly from Proposition 3.5 and Proposition 3.6.

4. When weakly injective modules are weakly projective.

LEMMA 4.1. *Let R be a ring such that every weakly injective right R-module has a projective cover. Then R is semiperfect.*

PROOF. Let S be a simple right R-module. By Lemma 2.6, there exists an infinite cardinal $\aleph$ such that $S \oplus E(S^{(\aleph)})$ is weakly injective. Therefore, by our hypothesis, $S \oplus E(S^{(\aleph)})$ has a projective cover. Our hypothesis also implies that $E(S^{(\aleph)})$ is weakly projective and therefore, it has a projective cover. Consequently, S has a projective cover. Thus R is semiperfect.

PROPOSITION 4.2. *Let R be a ring such that every weakly injective right R-module is weakly projective. Then R is right self injective with a finitely generated and essential socle containing a copy of each simple right module, i.e. R is a right* PF*-ring.*

PROOF. By Lemma 4.1, R is semiperfect. So we may write $R = \oplus\sum_{i=1}^{n} e_iR$, where $A = \{e_iR : i = 1, \ldots, k\}$ is a complete set of representatives for the indecomposable projective right R-modules. Then $B = \{\frac{e_iR}{e_iJ} : i = 1, \ldots, k\}$ is a complete set of representatives for the simple right R-modules. By the hypothesis and Lemma 2.10(2), each indecomposable injective right R-module is projective. In particular, for each simple right R-module S, the injective hull $E(S)$ is isomorphic to eR for some idempotent e in R. Therefore, every simple right module embeds in R. Define $f: B \longrightarrow A$ by $f(S) = e_iR$, where $e_iR \cong E(S)$. Clearly, f is one-to-one and therefore onto. Hence each indecomposable projective right module is injective and has a nonzero socle. Consequently, $R = \oplus\sum_{i=1}^{n} e_iR$ is injective and the right socle of R is finitely generated and essential, containing a copy of each simple right R-module. Thus R is a right PF-ring.

PROPOSITION 4.3. *Let R be a ring such that every weakly injective right R-module is weakly projective. Then R is a right and left perfect ring.*

PROOF. We will prove first that R is left perfect. By Lemma 4.1, R is semiperfect. Thus we may write $R = \oplus\sum_{i=1}^{n} e_iR$, where $\{e_iR : i = 1, \ldots, k\}$ is a complete set of representatives for indecomposable projective modules. Let M be a right module. Then by hypothesis, $E(M)$ is weakly projective having a projective cover $P\big(E(M)\big) = \oplus\sum_{i=1}^{k}(e_iR)^{(\alpha_i)}$. By Lemma 2.10(2), e_iR is a summand of $E(M)$ for each i such that $\alpha_i \neq 0$. Hence $\mathrm{soc}(e_iR) \subseteq \mathrm{soc}\big(E(M)\big) = \mathrm{soc}(M)$. Thus by Proposition 4.2 each nonzero right module has a nonzero socle and therefore R is left perfect.

Now, let N be a right module. Since soc(N) $\subseteq' N$, Lemma 2.2(2) and Lemma 2.6 yield that $N \oplus E(N^{(\aleph)})$ is weakly injective for some infinite cardinal $\aleph$. Therefore, by our hypothesis, $N \oplus E(N^{(\aleph)})$ and $E(N^{(\aleph)})$ are weakly projective. This implies $N \oplus E(N^{(\aleph)})$ and $E(N^{(\aleph)})$ have projective covers, yielding that N has a projective cover. Consequently, R is a right perfect ring.

LEMMA 4.4. *Let R be a ring such that every weakly injective right module is weakly projective and let eR be an indecomposable projective right R-module. Then for any right ideal I,* soc$(\frac{eR}{eI}) \cong [\frac{eR}{eJ}]^{(\alpha)}$ *for some cardinal* α.

PROOF. First we will show that soc(eR) $\cong$ (eR/eJ). By Lemma 2.6 and our hypothesis, there exists an infinite cardinal $\aleph$ such that $eR/eJ \oplus E\big((eR/eJ)^{(\aleph)}\big)$ is weakly projective. Since $eR \subseteq^{\oplus} P\Big(eR/eJ \oplus \big(E(eR/eJ)^{\aleph}\big)\Big)$, Lemma 2.10(2) yields that $eR \subseteq^{\oplus} eR/eJ \oplus E\big((eR/eJ)^{(\aleph)}\big)$. Therefore, soc($eR$) $\subseteq^{\oplus}$ soc$\Big(eR/eJ \oplus E\big((eR/eJ)^{\aleph}\big)\Big)$ and thus soc(eR) $\cong$ (eR/eJ).

Next we will show that $\frac{eR}{eJ}$ appears in the socle of every nonzero homomorphic image eR/eI of eR. Since $eR = P(\frac{eR}{eI})$, $eR \subseteq^{\oplus} P\big[\frac{eR}{eI} \oplus E(\frac{eR}{eI})^{(\aleph)}\big]$. But $\frac{eR}{eI} \oplus E(\frac{eR}{eI})^{(\aleph)}$ is weakly injective, hence weakly projective, so $eR \subseteq^{\oplus} \frac{eR}{eI} \oplus E(\frac{eR}{eI})^{(\aleph)}$. Hence, soc($eR$) embeds in soc$\big(\frac{eR}{eI} \oplus E(\frac{eR}{eI})^{(\aleph)}\big)$. Thus, we conclude that $\frac{eR}{eJ}$ embeds in soc(eR/eI).

Finally we will show that no other simple module appears in soc(eR/eI). Let soc$(\frac{eR}{eI}) = [\frac{eR}{eJ}]^{(\alpha)} \oplus K$ where $\frac{eR}{eJ}$ does not embed in K and $K \neq 0$. Since $[eR/eJ]^{(\alpha)} \oplus K \subseteq' eR/eI$, K embeds essentially in $(eR/eI)/[eR/eJ]^{(\alpha)} \cong eR/eA$ for some submodule eA of eR. Therefore by the first part of the proof, soc(K) contains a copy of $\frac{eR}{eJ}$, a contradiction. Thus soc$[eR/eI] \cong [\frac{eR}{eJ}]^{(\alpha)}$ for some cardinal α.

THEOREM 4.5. *Let R be a ring such that every weakly injective right module is weakly projective. Then R is a finite direct sum of matrix rings over local (right and left) perfect right* PF*-rings.*

PROOF. Since R is semiperfect we may write $R = \oplus \sum_{i=1}^{n} e_i R$ as a direct sum of indecomposable right ideals. We first show that if $\varphi: e_i R \longrightarrow e_j R$ is any nonzero R-homomorphism then $e_i R \cong e_j R$. Let $\varphi: e_i R \longrightarrow e_j R$ be a nonzero homomorphism. Then $\frac{e_i R}{\operatorname{Ker}\varphi}$ embeds in $e_j R$. Therefore, by the previous lemma, $\frac{e_i R}{eJ}$ embeds in soc($e_j R$) $\cong e_j R/e_j J$. So it follows that $e_i R \cong e_j R$. Now let $[e_i R] = \sum e_j R$, where the summation runs over all j for which $e_j R \cong e_i R$. Renumbering if necessary we may write $R = [e_1 R] \oplus \cdots \oplus [e_k R]$ where $k \leq n$. By the first part of the proof, $[e_i R]$ is an ideal in R and so $R \cong M_{n_1}(e_1 R e_1) \oplus \cdots \oplus M_{n_k}(e_k R e_k)$ where n_i is the number of summands in $[e_i R]$.

REMARK 4.6. It would be interesting to know if the ring R in Theorem 4.5 is also left PF. If this were true, then R will be QF.

5. **Further results.** In this section we consider those rings over which every weakly injective right module is weakly projective and every weakly projective right module is weakly injective. Theorem 3.3 shows that this is the case over local QF-rings. Indeed, in Theorem 5.4, we show that these conditions characterize the direct sum of matrix rings over local QF-rings. We note that over the ring of integers Z, every weakly projective right module is weakly injective (this is true since the only Z-modules having a projective cover are the projective modules). However, the injective Z-module Q is not (weakly) projective.

THEOREM 5.1. *Let R be a semiperfect ring such that every injective right R-module is weakly projective and every projective right R-module is weakly injective. Then R is a* QF-*ring.*

PROOF. By hypothesis each indecomposable injective right R-module is weakly projective and hence projective. This implies any direct sum of indecomposable injective right R-modules is projective, and so by hypothesis the sum is also weakly injective. It follows then by Lemma 2.5, that R is a right q. f. d. ring.

We show now each cyclic right R-module is essentially embeddable in a projective module. Let C be a cyclic module. Since R is a q. f. d. ring, there exists a finite direct sum $U_1 \oplus \cdots \oplus U_n \subseteq' C$. Then $E(C) = E(U_1) \oplus \cdots \oplus E(U_n)$. Therefore, $E(C)$ is projective, proving that C is essentially embeddable in a projective module. Thus R is right artinian (Lemma 2.3). Therefore, by Lemma 2.8, R is right self-injective, and so R is a QF-ring.

REMARK 5.2. The proof of Theorem 5.1 shows that if R is a semiperfect q. f. d. ring and each injective right module is weakly projective then R is a QF-ring.

The next theorem characterizes QF-rings in terms of weak-injectivity and weak projectivity.

THEOREM 5.3. *Let R be a ring. Then the following conditions are equivalent:*

(1) R is a QF-*ring.*

(2) R is a left perfect ring such that every projective right module is weakly injective.

(3) R is a semiperfect right q.f.d. ring such that every injective right module is weakly projective.

(4) R is a semiperfect ring such that every injective right module is weakly projective and every projective right module is weakly injective.

PROOF. (2) ⇒ (1) follows by Theorem 3.1.

(3) ⇒ (1) follows by Remark 5.2.

(4) ⇒ (1) follows by Theorem 5.1.

(1) ⇒ (2), (3) and (4) are clear.

We conclude with a characterization of matrix rings over local QF-rings.

THEOREM 5.4. *Let R be a ring. Then the following are equivalent:*

(1) R is left perfect and every weakly projective right R-module is weakly injective.

(2) R is a direct sum of matrix rings over local QF-*rings.*

(3) *R is a* QF-*ring such that for any indecomposable projective right module eR and for any right ideal I,* $\mathrm{soc}(eR/eI) = (eR/eJ)^n$ *for some positive integer n.*

(4) *R is right artinian and every weakly injective right R-module is weakly projective.*

(5) *Every weakly projective right R-module is weakly injective and every weakly injective right R-module is weakly projective.*

PROOF. (1) $\Rightarrow$ (3) follows by Proposition 3.5.

(3) $\Rightarrow$ (2) follows by Theorem 3.6.

(2) $\Rightarrow$ (1), follows by Corollary 3.4.

(4) $\Rightarrow$ (2) follows by Theorem 4.5.

(2) $\Rightarrow$ (4) follows by Corollary 3.4.

(5) $\Rightarrow$ (1) follows by Proposition 4.3.

(2) $\Rightarrow$ (5) follows by Corollary 3.4.

REMARKS 5.5. (1) A question that remains open is whether one can replace the requirement of "left perfect" in condition (1) of Theorem 5.4 by "right perfect" or even weaken it to semiperfect.

(2) As we stated in section 3, it is an open question whether a two-sided perfect, one sided self-injective ring is QF. If this were true, the condition "right artinian" in (4) of Theorem 5.4 could be removed.

ACKNOWLEDGEMENT. The authors would like to thank the referee for his suggestions.

REFERENCES

1. A. H. Al-Huzali, S. K. Jain and S. R. López-Permouth, *Rings whose cyclics have finite Goldie dimension*, J. Algebra **153**(1992), 37–40.
2. ———, *On the relative weak injectivity of rings and modules, Non-commutative Ring Theory*, Springer-Verlag, New York, Heidelberg **1448**, 1990, 93–98.
3. F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Springer-Verlag, New York, Heidelberg, 1974.
4. John Clark and Dinh van Huynh, *When is a self-injective semiperfect ring Quasi-Frobenius?*, preprint.
5. C. Faith, *Algebra II, Ring Theory*, Springer-Verlag, New York, Heidelberg, 1976.
6. ———, *When self-injective rings are Quasi-Frobenius: A report on a problem*, Centre de Recerca Matemtica, Institut DàEstudis Catalans, Barcelona, preprint, 1990.
7. S. K. Jain and S. R. López-Permouth, *Rings whose cyclics are essentially embeddable in projectives*, J. Algebra **128**(1990), 257–269.
8. S. K. Jain, S. R. López-Permouth and S. Singh, *On a class of QI-rings*, Glasgow Math. J. **34**(1992), 75–81.
9. S. K. Jain, S. R. López-Permouth and M. A. Saleh, *Weakly projective modules*. In: Ring Theory: Proceedings OSU, Denison Conference, World Scientific Press, New Jersey, (1993), 200–208.
10. F. Kasch, *Modules and rings*, Academic press, London, New York, 1982.
11. S. R. López-Permouth, *Rings characterized by their weakly injective modules*, Glasgow Math. J. **34**(1992), 349–353.

12. K. Oshiro, *Lifting modules, extending modules and their applications to* QF-*rings*, Hokkaido Math. J. **13**(1984), 310–338.

13. L. E. T. Wu and J. P. Jans, *On quasi projectives*, Illinois J. Math. **11**(1967), 439–448.

Ohio University
Athens, Ohio 45701
U.S.A.

Ohio University
Athens, Ohio 45701
U.S.A.

Yamaguchi University
Yoshido, Yamaguchi
Japan

BirZeit University
BirZeit
West Bank