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On uniform dimensions of ideals in right nonsingular rings

S.K. Jain [a,*], T.Y. Lam [b,1], André Leroy [c,2]

[a] Department of Mathematics, Ohio University, Athens, OH 45701, USA
[b] University of California, Berkeley, CA 94720, USA
[c] Université d'Artois, 62307 Lens Cedex, France

Abstract

For any (S,R)-bimodule M, one can define an invariant $d(M)$ by taking the supremum of n for which there exists a direct sum of nonzero subbimodules $N = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ such that N is essential in M as a right R-submodule. This invariant is a sort of hybrid between the right uniform dimension and the 2-sided uniform dimension. In this paper, we study the ideal structure of a right nonsingular ring R terms of the ideal structure of $Q^r_{max}(R)$ by working with the invariant $d(I) = d({}_RI_R)$ for ideals $I \subset R$. The family $\mathscr{F}(R)$ of ideals I for which there exists an ideal $J \subset R$ with $I \oplus J \subset_e R_R$ is characterized in various ways, and for $I \in \mathscr{F}(R)$, the invariant $d(I)$ is related to the direct product decomposition of the ring $E(I_R)$ (injective hull) in $Q^r_{max}(R)$. It is shown that $d(I)$ is very well-behaved for the ideals $I \in \mathscr{F}(R)$ and various results are obtained on the relationship between $d(I)$, $u.\ \dim({}_RI_R)$ and $u.\ \dim(I_R)$.

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1. Introduction

For a right nonsingular ring R, the maximal right ring of quotients $Q^r_{\max}(R)$ is well known to be a von Neumann regular right self-injective ring. There is an extensive classical literature on the structure of such rings, starting with papers of Johnson, Utumi, Findlay-Lambek, and continued in the work of many others. However, not too much information seemed available in relating the structure of R to that of $Q^r_{\max}(R)$. In this paper, we shall contribute to this problem by studying the ideal theory of R in

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relation to the ideal theory of $Q^{r}_{max}(R)$. Since much is known about the latter, we hope thus to be able to get useful information on the former.

The beginning point of this investigation is a certain new notion of "dimension" for bimodules, which can be introduced quite generally as follows. Let R, S be two rings, and M be an (S,R)-bimodule. The usual two-sided uniform dimension u.dim$({}_SM_R)$ is defined to be the supremum of the set of integers n for which M contains a direct sum of n nonzero subbimodules. This dimension is not difficult to deal with since it can be interpreted as the uniform dimension of M as a right $R \otimes S^{op}$-module. Now we can define a closely related invariant, $d(M) = d({}_SM_R)$, by taking the supremum of the set of integers n for which there exists a direct sum of nonzero subbimodules $N := M_1 \oplus \cdots \oplus M_n$ such that N is essential in M as a right R-submodule. Of course, we are giving preference to the right side in making this definition, so $d(M)$ may be thought of as a sort of hybrid between the right uniform dimension and the two-sided uniform dimension.

There seems to be no way in which $d(M)$ can be interpreted as a 1-sided uniform dimension over a single ring. This makes it difficult to obtain general information about "d" on the full category of (S,R)-bimodules. In fact, some of the usual properties of uniform dimensions will definitely *not* hold for "d". For instance, it is fairly easy to come up with examples of bimodules $N \subseteq M$ such that $d(M)$ is finite, but $d(N)$ is infinite. Yet, there are other dimensional properties which might conceivably hold for "d". For instance, it would be desirable to answer the following questions:

(1.1) *For any (S,R)-bimodules M and M', is $d(M \oplus M') = d(M) + d(M')$?*

(1.2) *For any (S,R)-bimodule M such that $d(M) = \infty$, does there exist an infinite direct sum of nonzero subbimodules $N := M_1 \oplus M_2 \oplus \cdots$ such that N is essential in M as a right R-submodule?*

The answer to both of these questions would presumably depend on a suitable version of the "Steinitz Replacement Theorem". But unfortunately, such a theorem does not seem to be available for the invariant "d".

For any (S,R)-bimodule M, it is of interest to look at the following family of subbimodules:

$$\mathscr{F}(M) = \{N \subseteq M \colon N \oplus N' \subseteq_e M_R \text{ for some subbimodule } N' \subseteq M\}, \tag{1.3}$$

where the notation $N \oplus N' \subseteq_e M_R$ means that $N \oplus N'$ is essential in M as a right R-submodule. For the ring R, we get, in particular, a family of ideals $\mathscr{F}(R) := \mathscr{F}({}_RR_R)$. This family of ideals and their d-invariants $d(I) = d({}_RI_R)$ will be the main focus of the present work.

In the case of a *right nonsingular* ring R, we will show that the family of ideals $\mathscr{F}(R)$ can be characterized in many other ways (Theorem 3.5). One particularly important characterization is that $I \in \mathscr{F}(R)$ iff the injective hull $E(I_R)$ is an *ideal* in $Q^{r}_{max}(R)$. Another characterization for such ideals I turns out to be $I \cap I' = 0$, where I' denotes the left annihilator of I in R. This condition first appeared in Johnson's 1957 paper

[8, p. 524],[3] although Johnson did not seem to be aware of the full range of equivalent conditions in our Theorem 3.5. The subfamily $\mathscr{B}(R) \subseteq \mathscr{F}(R)$ consisting of ideals in $\mathscr{F}(R)$ which are right essentially closed in R_R has also appeared in Johnson's work, and was shown in [8, p. 529] to be a complete Boolean algebra. In fact, as we observe in (3.15)(2), $\mathscr{B}(R)$ is isomorphic to the complete Boolean algebra of the central idempotents in $Q^{\mathrm{r}}_{\max}(R)$.

For the ideals I in the family $\mathscr{F}(R)$ over a right nonsingular ring R, various alternative descriptions for the invariant $d(I)$ are given in (3.16). We see from these descriptions that, on $\mathscr{F}(R)$, "d" has many of the usual features of a uniform dimension, and that $d(I)$ is an interesting measure for the "size" of the ideals I in $\mathscr{F}(R)$.

In Section 4, the invariant $d(I)$ is related to the study of the decomposition of von Neumann regular right self-injective rings. Here, again, we assume that R is right nonsingular and $I \in \mathscr{F}(R)$. We show in Theorem 4.1 that a direct sum of ideals $\bigoplus_i I_i \subseteq_{\mathrm{e}} I_R$ leads to a direct product decomposition of the ring $E(I_R) \subseteq Q^{\mathrm{r}}_{\max}(R)$, and vice versa. In particular, in the case when $d(I) < \infty$, $d(I)$ turns out to be just the number of "prime components" of the von Neumann regular right self-injective ring $E(I_R)$, or alternatively, the number of "atoms" in the Boolean algebra of central idempotents in $E(I_R)$ (Theorem 4.5). Taking I to be R, the case when $Q^{\mathrm{r}}_{\max}(R)$ is a prime ring then corresponds to $d(R) = 1$: such R's are the *right irreducible rings* in the sense of R.E. Johnson. A partial list of characterizations for such rings is assembled (and briefly discussed) in Theorem 4.8.

A byproduct of the work in Section 4 is that both of the properties (1.1) and (1.2) are both confirmed for the ideals in the family $\mathscr{F}(R)$ over a right nonsingular ring R. In fact, contrary to the case of one-sided uniform dimension, one gets even the full dimension formula $d(I) + d(J) = d(I + J) + d(I \cap J)$ for $I, J \in \mathscr{F}(R)$.

For an ideal $I \subseteq R$, the invariant $d(I)$ is related to the one-sided and two-sided uniform dimensions of I by the inequality

$$d(I) \leq \text{u.dim}({}_R I_R) \leq \text{u.dim}(I_R). \tag{1.4}$$

In general, these three invariants are different. But there are various special classes of $I \subseteq R$ for which two or all three of them turn out to be the same. For instance, we show that the first two invariants in (1.4) are the same if I contains no nonzero nilpotent ideals of R. From this, we deduce that, for $I \in \mathscr{F}(R)$ over a right nonsingular ring R, the first two invariants in (1.4) are the same if the symmetric maximal quotient ring of R happens to be semiprime (Theorem 5.6). In particular, this applies to any *Utumi ring* R, that is, a right nonsingular ring R for which $Q^{\mathrm{r}}_{\max}(R) = Q^{\ell}_{\max}(R)$ (Corollary 5.7). It follows that, for such a ring with $n := \text{u.dim}({}_R R_R) < \infty$, the maximal right quotient ring $Q^{\mathrm{r}}_{\max}(R)$ will decompose into a direct product of exactly n simple self-injective von Neumann regular rings. Finally, it is also shown, in (5.10), that all three invariants in (1.4) are equal for any ideal I in a reduced right Utumi ring.

[3] Johnson referred to this property by saying that the "ring" (possibly without identity) I is a "left faithful ring".

2. Definitions and notations

Throughout this paper, we denote by $Q^{\mathrm{r}}_{\max}(R)$, $Q^{l}_{\max}(R)$ and $Q_{\sigma}(R)$, respectively the right, left and symmetric maximal quotient rings of a ring R. Here, the symmetric maximal quotient ring is defined as in [13]; namely,

$$Q_{\sigma}(R) = \{x \in Q^{\mathrm{r}}_{\max}(R) : Kx \subseteq R \text{ for some dense left ideal } K \subseteq R\}.$$

We write $M' \subseteq_{\mathrm{e}} M$ (resp. $M' \subseteq_{\mathrm{d}} M$) to denote the fact that the R-submodule M' is essential (resp. dense) in the R-module M. The injective hull of M will be denoted by $E(M)$, and the singular submodule of M will be denoted by $\mathscr{Z}(M)$. If it is necessary to indicate whether M is a right or left R-module, we shall do so by writing M_R or ${}_RM$.

The notation u.dim(M_R) (resp. u.dim$({}_SN)$) will be used throughout to denote the uniform dimension of a right module M_R (resp. a left module ${}_SN$). If M is an (S,R)-bimodule, we have also a two-sided uniform dimensi on u.dim$({}_SM_R)$, defined to be the uniform dimension of M as a right $R \otimes S^{\mathrm{op}}$-module (see [15, p. 53]). The invariant $d(M)$ for the bimodule ${}_SM_R$ (and the associated family $\mathscr{F}(M)$ of subbimodules of M) will be as defined in the Introduction. In general,

$$d(M) \le \text{u.dim}({}_SM_R) \le \min\{\text{u.dim}(M_R),\ \text{u.dim}({}_SM)\}. \tag{2.1}$$

Although the invariant $d(M)$ is defined somewhat in the spirit of the two-sided uniform dimension u.dim$({}_SM_R)$, we must exercise caution in working with "d" since it *does not* have all the usual properties of a uniform dimension on the full category of (S,R)-bimodules. Nevertheless, the invariant d is better behaved on the 2-sided ideals of a ring R, especially on those which constitute the family $\mathscr{F}(R)$ for a right nonsingular ring R. For the most part of this paper, we will be studying the d-invariant in this particular setting.

Throughout this paper, all rings have an identity element 1, and all modules are unital. The word "ideal" always means a two-sided ideal. For any subset A in a ring R, A^{l} shall denote the left annihilator of A in R. Note that A^{l} is always a left ideal in R, and if A itself is a left ideal, then A^{l} is an ideal in R. By A^{2}, we shall mean $(A^{l})^{l}$, etc.

For other standard notations, terminology and basic facts for rings and modules used in this paper, the reader is referred to the classical books [3, 4, 15, 16].

3. The families of ideals $\mathscr{F}(R)$ and $\mathscr{B}(R)$

In this section, we develop the basic results on uniform dimensions to be used in the rest of the paper, and introduce the families of ideals $\mathscr{F}(R)$ and $\mathscr{B}(R)$ in a ring R. In Sections 3 and 4, these families will be studied mostly over a right nonsingular ring R.

Our first lemma is possibly folklore in the theory of nonsingular modules. We include it here with a full proof since there is no convenient reference for it in the literature.

Lemma 3.1. *Let R be a subring of a ring S such that $R_R \subseteq_e S_R$, and let $N \subseteq M$ be right S-modules.*

(1) *If M_R is nonsingular, then $N_S \subseteq_d M_S$ iff $N_R \subseteq_d M_R$;*

(2) *If N_R is nonsingular, then $N_S \subseteq_e M_S$ iff $N_R \subseteq_e M_R$.*

Proof. (1) The "if" part is trivial (and is true without any assumptions on M or on $R \subseteq S$). For the "only if" part, assume that $N_S \subseteq_d M_S$, and let $x, y \in M$ with $x \neq 0$. There exists $s \in S$ such that $xs \neq 0$ and $ys \in N$. Since $R_R \subseteq_e S_R$, $sK \subseteq R$ for some right ideal $K \subseteq_e R_R$. Now $xs \notin \mathscr{Z}(M_R) = 0$, so $(xs)k \neq 0$ for some $k \in K$. For $r = sk \in R$, we have $xr \neq 0$ and $yr = (ys)k \in N\,K \subseteq N$. This shows that $N_R \subseteq_d M_R$.

(2) For the "only if" part in (2), repeat the argument above with $y = x \neq 0$. Here $0 \neq xs = ys \in N$, so the (weaker) assumption $\mathscr{Z}(N_R) = 0$ would have sufficed for the argument. The "if" part is trivial as before. □

Lemma 3.2. *Let R be a right nonsingular ring, and $Q = Q^{\mathrm{r}}_{\max}(R)$. For any right ideal $I \subseteq R$, the injective hull $E(I_R)$ (formed in Q_R) is a right ideal in Q. If I is an ideal, then $E(I_R)$ is an (R, Q)-subbimodule of Q.*

Proof. First note that, since R is right nonsingular and $R \subseteq_e Q_R$, Q_R is a right nonsingular module. In this situation, it is well known that I_R has a unique essential closure in Q_R given by

$$C := \{c \in Q\colon\ cK \subseteq I \text{ for some essential right ideal } K \subseteq R\}.$$

In particular, $E(I) = C$. (Here and in the following, $E(I)$ shall *always* mean $E(I_R)$.) From the above equation for C, it is an easy exercise to check that C is a right ideal in Q. In case I is an ideal in R, the same equation for C implies that it is also a left R-submodule of Q. □

Remark. In the special case when $\mathrm{u.dim}(R_R) < \infty$, it is known that $E(I) = I \cdot Q$ (see, for instance, [6, Exer. 4ZK(b), p. 84]). In this case, it is immediately clear that $E(I)$ is an (R, Q)-bimodule. However, in the general case, one has only $I \cdot Q \subseteq E(I)$.

In general, if I is a right (or even 2-sided) ideal in R, $E(I)$ may not be an ideal in Q. We shall now proceed to find the necessary and sufficient conditions for $E(I)$ to be an ideal.

Proposition 3.3. *Let R be a right nonsingular ring, $Q = Q^{\mathrm{r}}_{\max}(R)$, and $A_1, \ldots, A_n$ be right ideals in R such that $\bigoplus_{i=1}^{n} A_i \subseteq_e R_R$. If the A_i are mutually orthogonal (i.e. $A_iA_j = 0$ whenever $i \neq j$), then each injective hull $E(A_i)$ is an ideal in Q, with $\bigoplus_{i=1}^{n} E(A_i) = Q$.*

Proof. Taking injective hulls with $\bigoplus_{i=1}^{n} A_i \subseteq_e R_R$, we have $\bigoplus_{i=1}^{n} E(A_i) = Q$. Since each $E(A_i)$ is a right ideal in Q by (3.2), we can write $E(A_i) = e_iQ$ where $e_1, \ldots, e_n$ are mutually orthogonal idempotents in Q with sum 1. In the rest of the proof, we will show that each e_i is a *central* idempotent in Q. Certainly, this will imply that each $E(A_i) = e_i Q$ is an ideal in Q.

As a first step, we claim that $A_i \cdot E(A_j) = 0$ whenever $i \neq j$. Indeed, for $a \in A_i$ and $b \in E(A_j)$, we have $bK \subseteq A_j$ for some right ideal $K \subseteq_e R_R$. Therefore, $a \cdot bK \subseteq A_i A_j = 0$. Since Q_R is nonsingular, this implies that $ab \in \mathscr{Z}(Q_R) = 0$, which proves our claim.

Next we claim that e_i *commutes with each element in* A_j, *for all* i, j. Indeed, let $a \in A_j$. If $j \neq i$, then $e_i a \in e_i e_j Q = 0$, and $ae_i \in A_j \cdot E(A_i) = 0$ (by the last paragraph). Now assume $j = i$. Then $a \in A_i \subseteq e_i Q$ implies that

$$e_i a = a = a\left(e_i + \sum_{k \neq i} e_k\right) = ae_i,$$

since $ae_k \in A_i \cdot E(A_k) = 0$ for any $k \neq i$.

We have now shown that each e_i commutes elementwise with the direct sum $A_1 \oplus \cdots \oplus A_n$. Since

$$A_1 \oplus \cdots \oplus A_n \subseteq_e R_R \subseteq_e Q_R,$$

a standard argument using the nonsingularity of Q_R shows that each e_i is central in Q, as desired. □

Remark 3.4. Note that the proposition above is applicable to any finite direct sum of *ideals* $\bigoplus_{i=1}^{n} A_i \subseteq_e R_R$, since, in this case, the A_i's are automatically mutually orthogonal.

With the help of the above lemma, we can now formulate the conditions for an injective hull $E(I)$ ($I_R \subseteq R$) to be an ideal in Q.

Theorem 3.5. *Let R and Q be as in* (3.3), *and I be a right ideal of R. The following statements are equivalent*:

(1) *$E(I)$ is an ideal in Q*;

(2) *$E(I) = eQ$ where e is a central idempotent in Q*;

(3) *There exists a right ideal P' in Q orthogonal to $E(I)$ such that $E(I) \oplus P' = Q$*;

(4) *There exists a right ideal P' in Q orthogonal to $E(I)$ such that $E(I) \oplus P' \subseteq_e Q_R$*;

(5) *There exists a right ideal J in R orthogonal to I such that $I \oplus J \subseteq_e R_R$*;

(6) *There exists an ideal P in Q such that $I \subseteq_e P_R$.*

Proof. (2) ⇒ (3) Just take P' to be the ideal $(1-e)Q$.

(3) ⇒ (4) is a tautology.

(4) ⇒ (5) Suppose the ideal P' exists as in (4). Since $R \subseteq_e Q_R$, we have $P' \cap R \subseteq_e P'$. Together with $I \subseteq_e E(I)_R$, this shows that

$$I \oplus (P' \cap R) \subseteq_e E(I) \oplus P' \subseteq_e Q_R.$$

Thus, $I \oplus J \subseteq_e R_R$ for the right ideal $J := P' \cap R$. Since P' is orthogonal to $E(I)$, J is orthogonal to I.

(5) ⇒ (1) This follows from (3.3) in the special case $n = 2$.

(1) ⇒ (6) Since $E(I)$ is an ideal by assumption, we can take P in (6) to be $E(I)$.

(6) ⇒ (2) Since P is an ideal in Q, there exists, by [4, (9.5)], a central idempotent $e \in Q$ such that $P \subseteq_e (eQ)_Q$. (This is an easy result. In fact, we shall prove it in a slightly more general context in (3.14)(3) below.) By (3.1)(2), we have $P \subseteq_e (eQ)_R$, and, together with $I \subseteq_e P_R$, this implies that $I \subseteq_e (eQ)_R$. Since $(eQ)_R$ is injective, we have $E(I) = eQ$ as desired. □

Remark 3.6. Note that the arguments given above would have worked if the $P' \subseteq Q$ in (3) or (4) is assumed to be an R-submodule of Q_R, instead of a right ideal in Q. Therefore, we could have added two more equivalent statements (3*) and (4*) to (3.5), by changing the condition that $P' \subseteq Q$ be a right ideal to P' being an R-submodule of Q_R. More significantly, in the case when I is an *ideal* of R (and R is right nonsingular), we can also add two more equivalent conditions:[4]

(5*) *There exists an ideal J in R such that $I \oplus J \subseteq_e R_R$.*

(7) $I \cap I' = 0$.

Indeed, (5*) ⇒ (5) follows from Remark 3.4. For (5) ⇒ (7), let J be as in (5) (we shall only need the properties $IJ = 0$ and $I + J \subseteq_e R_R$), and consider any $x \in I \cap I'$. Then $xI = 0$ and $xJ = 0$, so $x \cdot (I + J) = 0$. Since $I + J \subseteq_e R_R$, $x \in \mathscr{Z}(R_R) = 0$. Finally, for (7) ⇒ (5*), let B be a right ideal complement to I_R in R_R. Then $I \oplus B \subseteq_e R_R$. Now $BI \subseteq B \cap I = 0$, so $B \subseteq I'$. For the ideal $J := I'$, we have then $I \oplus J \subseteq_e R_R$, since $I \oplus J \supseteq I \oplus B$. (In particular, we have $B = I'$, and this is the *unique* complement to I in R_R.)

Note that, among all conditions given above, (5*) is the only one with the following two features: (A) it involves only the ring R, and not its maximal right ring of quotients Q; and (B) it can be formulated purely in the language of bimodules. This prompts the following general formulation.

Definition 3.7. For any rings R, S and any (S, R)-bimodule M, let $\mathscr{F}(M)$ be the set of subbimodules $I \subseteq M$ for which there exists a subbimodule $J \subseteq M$ such that $I \oplus J \subseteq_e M_R$. For any ring R, we write $\mathscr{F}(R)$ for $\mathscr{F}({}_RR_R)$, that is, the set of ideals in R satisfying the condition (5*) above.

Of course, in the case when R is right nonsingular and $Q = Q^r_{max}(R)$, the ideals I in $\mathscr{F}(R)$ are characterized by *any* of the conditions in (3.5) and (3.6). The notation $\mathscr{F}(R)$ follows Johnson [8, p. 524], who used condition (7) as its definition, but did not seem to realize the full range of equivalent conditions in (3.5) and (3.6). Note

[4] For the condition (7) below, recall that I' denotes the left annihilator of I, and I'' means $(I')'$.

that, since $R \subseteq_e Q_R$, *the family* $\mathscr{F}(R)$ *includes the contractions of all ideals of* Q *to* R.

We could have introduced also the set $\mathscr{F}_r(R)$ of *right* ideals $I \subseteq R$ satisfying the condition (5). In the case when R is right nonsingular, it is easy to see that a right ideal I belongs to $\mathscr{F}_r(R)$ iff I is right essential in some ideal belonging to $\mathscr{F}(R)$, iff $I \subseteq_e (RI)_R$ and $RI \in \mathscr{F}(R)$. Therefore, questions about $\mathscr{F}_r(R)$ can often be reduced to questions about $\mathscr{F}(R)$. For this reason, we shall pass up on the family $\mathscr{F}_r(R)$ in the rest of the paper, and just focus our attention on the family $\mathscr{F}(R)$ (mostly over right nonsingular rings R).

For R and Q as in (3.3), we record the following useful consequence of (3.5).

Corollary 3.8. *Let S be any ring between R and Q. Then we have a mapping* $^*: \mathscr{F}(R) \to \mathscr{F}(S)$ *defined by* $I \mapsto I^* := SIS$ *for any* $I \in \mathscr{F}(R)$.

Proof. It is well known that S is also a right nonsingular ring, with $Q^r_{\max}(S) = Q$. For $I \in \mathscr{F}(R)$, we only have to make sure that $I^* \in \mathscr{F}(S)$. This follows easily by checking the condition (6) in (3.5): if $I \subseteq_e P_R$ for some ideal P of Q, then we also have $I^* = SIS \subseteq_e P_R$, and hence $I^* \subseteq_e P_S$. □

At this time, let us introduce two more pieces of notations.

(3.9) *For any ring R, we write $B(R)$ for the set of all central idempotents in R.* It is well known that, with respect to the standard partial ordering and binary join/meet operations for central idempotents, $B(R)$ is a Boolean algebra. It is often convenient to think of $B(R)$ as the Boolean algebra of ideals eR for e ranging over $B(R)$.

(3.10) *For any ring R, we write $\mathscr{B}(R)$ for the set of all ideals in $\mathscr{F}(R)$ which are right essentially closed in R_R.* In the case when R is right nonsingular, it is easy to show that

$$\mathscr{B}(R) = \{I \in \mathscr{F}(R): I = I''\}, \tag{3.11}$$

using the fact that, for $I \in \mathscr{F}(R)$, I', I'' are both complements in R_R (and are hence right essentially closed in R_R). In the form (3.11) (for right nonsingular rings R), the family $\mathscr{B}(R)$ was first introduced by Johnson [8, p. 542], who denoted it by $\mathscr{F}''(R)$, and showed that it is the "center" of the lattice $\mathscr{F}(R)$. Note that there are two natural maps

$$c: \mathscr{F}(R) \to \mathscr{B}(R) \quad \text{and} \quad \ell: \mathscr{F}(R) \to \mathscr{B}(R), \tag{3.12}$$

defined by sending $I \in \mathscr{F}(R)$ respectively to I^c (the unique right essential closure of I in R) and I' (the left annihilator of I in R). The map c is easily seen to be a "closure operator" in the sense of [16, III.7].

Remark 3.13. For *semiprime* (but not necessarilly right nonsingular) rings R, the two families $\mathscr{F}(R)$ and $\mathscr{B}(R)$ are particularly easy to identify. In fact, for any ideal I in a semiprime ring R, $I \cap I'$ is an ideal of square zero, so $I \cap I' = 0$. Since the implication (7) $\Rightarrow$ (5*) in (3.6) holds for any ring, we have $I \in \mathscr{F}(R)$. Thus $\mathscr{F}(R)$ is the family of *all* ideals in R, and it follows that $\mathscr{B}(R)$ is the family of all ideals which are right essentially closed in R.

Recall that a *Baer ring* is a ring in which every left (equivalently, right) annihilator ideal is generated by an idempotent. It is well known that any Baer ring is a (left and right) nonsingular ring, and any right self-injective von Neumann regular ring is a semiprime Baer ring. For semiprime Baer rings, we have the following result.

Proposition 3.14. *Let R be a semiprime Baer ring, with a maximal right quotient ring Q. Then*

(1) *The map $\Psi : B(R) \to \mathscr{B}(R)$ defined by $\Psi(e) = eR$ for every $e \in B(R)$ is a bijection.*

(2) *$B(R) = B(Q)$, and these are complete Boolean algebras.*

(3) *Any ideal $I \subseteq R$ is right essential in eR for some $e \in B(R)$.*

Proof. (1) Clearly Ψ is injective, so it suffices to show that Ψ is also *surjective*. Consider any $I \in \mathscr{B}(R)$. By (3.11), $I = I''$. In particular, I is a left annihilator, so $I = Re$ for some idempotent $e \in R$. Since R is semiprime and I is ideal, it follows from [3, (2.33)] that e is a *central* idempotent. Hence $I = eR = \Psi(e)$. This shows that Ψ is a bijection.

(2) Note first that any element in the center of R is also in the center of Q (cf. the end of the proof of (3.3)). Therefore, we have an inclusion $B(R) \subseteq B(Q)$. To see that this an equality, let $e \in B(Q)$. Then the ideal $I := eQ \cap R$ is right essentially closed in R, so by (3.13) $I \in \mathscr{B}(R)$. By (1), $I = e_0R$ for some $e_0 \in B(R)$. Since $(e_0Q)_R$ is an R-direct summand of $(eQ)_R$, and $E(I) = eQ$, we must have $eQ = e_0Q$, and so $e = e_0 \in B(R)$. Finally, $\{eR : e \in B(R)\}$ is just the family of all annihilator ideals in the semiprime ring R, so it is closed with respect to arbitrary intersections. By [16, Proposition III.1.2], this implies that $B(R)$ is a complete Boolean algebra.

(3) Let I be any ideal in R. By (3.13), we have $I \in \mathscr{F}(R)$, so by (3.12), the unique right essential closure I^c of I in R belongs to $\mathscr{B}(R)$. By (1), $I^c = eR$ for some $e \in B(R)$, so we have $I \subseteq_e (eR)_R$. □

The conclusions (2) and (3) above are well known in the case when R is a right self-injective von Neumann regular ring; see, respectively, [4, (9.9)] and [4, (9.5)]. Here, we have proved them more generally for any semiprime Baer ring R.

Returning now to general right nonsingular rings, we collect in the following proposition a few key properties of $\mathscr{F}(R)$ and $\mathscr{B}(R)$. The first of these has already appeared in [8, Theorem (2.4)].

Proposition 3.15. *Let R be a right nonsingular ring, and $Q = Q^{r}_{\max}(R)$. Then*

(1) The family of ideals $\mathscr{F}(R)$ is closed with respect to arbitrary sums and finite intersections. With respect to the standard partial ordering given by inclusion, $\mathscr{F}(R)$ forms a complete lattice.

(2) There is a one-one correspondence $\Phi : B(Q) \to \mathscr{B}(R)$ given by $\Phi(e) = eQ \cap R$ for $e \in B(Q)$, and $\Phi^{-1}(I) = e$ for $I \in \mathscr{B}(R)$, where $E(I) = eQ$. With respect to inclusion again, $\mathscr{B}(R)$ is a complete Boolean algebra where, for an arbitrary set of ideals $\{I_i\} \subseteq \mathscr{B}(R)$, the meet of $\{I_i\}$ is given by $\bigcap_i I_i$, and the join of the same is given by the right essential closure of $\sum_i I_i$ in R.

Proof. (1) Note that if $\{I_i\colon i \in C\} \subseteq \mathscr{F}(R)$, say with $I_i \subseteq_e (P_i)_R$ where the P_i's are ideals in Q, then $\sum_i I_i \subseteq_e (\sum_i P_i)_R$, and $\bigcap_i I_i \subseteq_e (\bigcap_i P_i)_R$ in case $|C| < \infty$. This checks the first statement of (1), and it follows from [16, Proposition III.1.2] that $\mathscr{F}(R)$ is a complete lattice. (The meet of $\{I_i\colon i \in C\}$ for arbitrary C is the sum of ideals in $\bigcap_i I_i$ belonging to $\mathscr{F}(R)$.)

(2) First, the fact that Φ is a one–one correspondence follows from (3.5). Second, since Q is a semiprime Baer ring, we know from (3.14)(2) that $B(Q)$ is a *complete* Boolean algebra. In fact, for any family $\{e_i\} \subseteq B(Q)$, the meet and the join of $\{e_i\}$ are defined via the equations $(\bigwedge e_i)Q = \bigcap e_i Q$ and $(\bigvee e_i)Q = E((\sum_i e_i Q)_Q)$ (see also [4, (9.9)]). Using these characterizations and the one–one correspondence Φ above, it is then easy to check that, with respect to the partial ordering given by inclusion, $\mathscr{B}(R)$ is also a complete Boolean algebra, with the meet and the join as described in (2). $\square$

Next, we would like to give some alternative descriptions for the invariant $d(I)$ for the ideals $I \in \mathscr{F}(R)$, in case R is a right nonsingular ring.

Proposition 3.16. *Let R, Q be as in (3.3), and $I \in \mathscr{F}(R)$. Let $m\,(\leq \infty)$ be the supremum of the set*

$$\{n \in \mathbb{N}\colon \text{there exist nonzero mutually orthogonal right ideals } A_1, \ldots, A_n \subseteq I \text{ such that } A_1 \oplus \cdots \oplus A_n \subseteq_e I_R\},$$

and $m'\,(\leq \infty)$ be the supremum of the set

$$\{k \in \mathbb{N}\colon \text{there exist nonzero ideals } I_1, \ldots, I_k \in \mathscr{F}(R) \text{ such that } I_1 \oplus \cdots \oplus I_k \subseteq I\}.$$

Then $d(I) = m = m'$.

Proof. Let us show that $d(I) \leq m \leq m' \leq d(I)$. For the first inequality, let $I_1 \oplus \cdots \oplus I_t$ be any direct sum of nonzero ideals in I which is essential in I_R. Then the I_i's are mutually orthogonal by (3.4). Thus, we have $t \leq m$, and so $d(I) \leq m$. To see that $m \leq m'$, suppose the right ideals $A_1, \ldots, A_n$ are as in the definition of m. Since $I \in \mathscr{F}(R)$, we have (by the equivalent conditions discussed in (3.6)):

$$A_1 \oplus \cdots \oplus A_n \oplus I' \subseteq_e (I \oplus I')_R \subseteq_e R_R.$$

Here, for any i, $I' A_i = 0$, and $A_i I' \subseteq A_i \cap I' = 0$. Therefore, by (3.3), each $E(A_i)$ is an ideal in Q, and so $E(A_i) \cap I \in \mathscr{F}(R)$. Recalling that $\mathscr{F}(R)$ is closed under (finite) intersections, we see that

$$I_i := E(A_i) \cap I = (E(A_i) \cap R) \cap I \in \mathscr{F}(R).$$

Now $\sum_{i=1}^{n} E(A_i)$ is automatically a direct sum, so we have $\bigoplus_{i=1}^{n} I_i \subseteq I$. This shows that $n \leq m'$, and so $m \leq m'$. Finally, to see that $m' \leq d(I)$, let $\{I_i\colon 1 \leq i \leq k\}$ be as in the definition of m'. By (3.15)(1), $J := I_1 \oplus \cdots \oplus I_k \in \mathscr{F}(R)$, so $J \oplus J' \subseteq_e R_R$. Taking intersection of both sides with I, we get

$$I_1 \oplus \cdots \oplus I_k \oplus (I \cap J') = J \oplus (I \cap J') \subseteq_e I_R.$$

This shows that $k \leq d(I)$ (noting that $I \cap J'$ is possibly zero), and consequently $m' \leq d(I)$. □

Remark 3.17. Actually, in the context of (3.16), there is yet another description of $d(I)$. Using either $d(I) = m$ or $d(I) = m'$, one can show that $d(I)$ is also the supremum of the set of integers r for which there exists a chain

$$0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_r \subseteq I$$

such that $B_i \in \mathscr{F}(R)$ for all i, and each B_i $(1 \leq i < r)$ is *not* right essential in B_{i+1}. The proof of this is left as an exercise to the reader.

For later reference, we shall prove here a general result on the behavior of the "d"-invariant and the 1-sided and 2-sided uniform dimensions vis-à-vis the change of rings. For part (4) below, recall that $Q_\sigma(R)$ denotes the symmetric maximal ring of quotients of R.

Theorem 3.18. *Let R, Q be as in (3.3), and S be any subring of Q containing R. For any ideal $I \in \mathscr{F}(R)$, let $I^* := SIS$ be the ideal generated by I in S. Then*

(1) $d(I) = d(I^*)$ (*where $d(I^*)$ is supposed to mean $d({}_S I^*_S)$*).

(2) $\text{u.dim}\,(I_R) = \text{u.dim}\,(I^*_R) = \text{u.dim}\,(I^*_S)$.

(3) $\text{u.dim}\,({}_R I_R) = \text{u.dim}\,({}_R I^*_R) = \text{u.dim}\,({}_R I^*_S) \geq \text{u.dim}\,({}_S I^*_S)$.

(4) *Equality holds throughout in* (3) *if* $S \subseteq Q_\sigma(R)$.

Proof. (1) Since $I \in \mathscr{F}(R)$, $E(I)$ is an ideal in Q. Hence $I^* = SIS \subseteq E(I)$, and so $I \subseteq_e I^*_R$. If $I_1 \oplus \cdots \oplus I_n \subseteq_e I_R$ where the I_i's are nonzero ideals in R, then these are mutually orthogonal, and as in the proof of (3.16), the $E(I_i)$'s are also ideals in Q, with $\bigoplus_i E(I_i) = E(I)$. Since the direct sum $\bigoplus_i (E(I_i) \cap I^*)$ contains $\bigoplus_i I_i \subseteq_e I^*_R$, it is essential in I^*_R, and hence also essential in I^*_S. Therefore, $n \leq d(I^*)$, and we have

$d(I) \leq d(I^*)$. To prove the reverse inequality, let $J = J_1 \oplus \cdots \oplus J_n$ be a direct sum of nonzero ideals in the ring S such that $J \subseteq_e I_S^*$. Since J_R is a nonsingular R-module, we have $J \subseteq_e I_R^*$ by (3.1)(2). On the other hand, $I \subseteq_e I_R^*$ implies that $J_i \cap I \subseteq_e (J_i)_R$. Taking direct sums leads to $\bigoplus_i (J_i \cap I) \subseteq_e J_R \subseteq_e I_R^*$, so *a fortiori* $\bigoplus_i (J_i \cap I) \subseteq_e I_R$. Since each $J_i \cap I \neq 0$, this shows that $n \leq d(I)$, and so $d(I^*) \leq d(I)$.

(2) Here again, we exploit the fact that $I \subseteq_e I_R^*$. This implies that any nonzero R-submodule of I_R^* intersects I at a nonzero right ideal of R. From this, we see easily that $\text{u.dim}(I_R) \geq \text{u.dim}(I_R^*) \geq \text{u.dim}(I_S^*)$. Thus it remains only to show that $\text{u.dim}(I_S^*) \geq \text{u.dim}(I_R)$. Consider any direct sum of nonzero right ideals $\bigoplus_{i=1}^n A_i \subseteq I$. This gives a direct sum $\bigoplus_{i=1}^n E(A_i) \subseteq Q$, and so the sum $\sum_{i=1}^n A_i S \subseteq \bigoplus_{i=1}^n E(A_i)$ is also direct. From $\bigoplus_{i=1}^n A_i S \subseteq I^*$, we have then $n \leq \text{u.dim}(I_S^*)$, and so $\text{u.dim}(I_R) \leq \text{u.dim}(I_S^*)$.

(3) As in the proof of (2), we have

$$\text{u.dim}({}_R I_R) \geq \text{u.dim}({}_R I_R^*) \geq \text{u.dim}({}_R I_S^*) \geq \text{u.dim}({}_S I_S^*).$$

since any nonzero (R,R)-subbimodule of I^* intersects I at a nonzero ideal of R. Thus it remains only to show that $\text{u.dim}({}_R I_S^*) \geq \text{u.dim}({}_R I_R)$. Consider any direct sum of nonzero ideals $\bigoplus_{i=1}^n I_i \subseteq I$. Then we have a direct sum $\bigoplus_{i=1}^n E(I_i) \subseteq Q$, where, by Lemma 3.2, each $E(I_i)$ is an (R,Q)-subbimodule in Q. It follows that $\bigoplus_{i=1}^n (E(I_i) \cap I^*) \subseteq I^*$ is a direct sum of nonzero (R,S)-subbimodules in I^*. This clearly implies that $\text{u.dim}({}_R I_S^*) \geq \text{u.dim}({}_R I_R)$.

(4) Now suppose $S \subseteq Q_\sigma(R)$. It suffices to prove that $\text{u.dim}({}_R I_R) \leq \text{u.dim}({}_S I_S^*)$. Let $I_1 \oplus \cdots \oplus I_n \subseteq I$ be a direct sum of nonzero ideals in R. We are done if we can show that $\sum_{i=1}^n S I_i S \subseteq I^*$ is a *direct* sum in S. Suppose $\sum_i x_i = 0$, where $x_i \in S I_i S$. Let us write $x_i = \sum_j s'_{ij} x_{ij} s_{ij}$, where $x_{ij} \in I_i$, and $s_{ij}, s'_{ij} \in S$. Since the intersection of a finite number of dense right (resp. left) ideals is dense, there exist a right ideal $J \subseteq_d R_R$ and a left ideal $J' \subseteq_d {}_R R$ such that $s_{ij} J \subseteq R$ and $J' s'_{ij} \subseteq R$ for all i, j. Then $\sum_i \sum_j (J' s'_{ij}) x_{ij} (s_{ij} J) = 0$ shows that

$$\sum_j (J' s'_{ij}) x_{ij} (s_{ij} J) = 0 \quad \text{for all } i. \tag{3.19}$$

Now by the transitivity of denseness, $J \subseteq_d R_R \subseteq_d S_R$ implies $J \subseteq_d S_R$, so J has zero left annihilator in S (see [13, (1.1)(iii)]). Similarly, $J' \subseteq_d {}_R S$ and J' has zero right annihilator in S. Therefore, (3.19) implies that $x_i = \sum_j s'_{ij} x_{ij} s_{ij} = 0$ for all i, as desired. □

Remark 3.20. In general, in (3.18)(3) above, the (last) inequality may not be an equality, even for $S = Q^r_{\max}(R)$ and $I = R$; see (3.22). However, in the case when $Q^r_{\max}(R) = Q^l_{\max}(R)$ (and R is a right nonsingular ring), it will follow from (5.7) below that *all* the invariants listed in (1) and (3) of (3.18) are equal.

We shall conclude this section with a couple of examples.

Example 3.21. If R is a prime ring, clearly $d(R) = \text{u.dim}({}_RR_R) = 1$. More generally, if $R = R_1 \times \cdots \times R_n$ where the R_i's are prime rings, then by decomposing ideals of R into their components in the R_i's, it is easy to show that $d(R) = \text{u.dim}({}_RR_R) = n$. If, on the other hand, R is a direct product of an infinite number of nonzero rings R_i, then we have $\bigoplus R_i \subseteq_e R_R$, and hence $d(R) = \text{u.dim}({}_RR_R) = \infty$. These observations will be crucial to the work in the next section.

Example 3.22. Let $F \subseteq K$ be fields with $n := \dim_F K \in \mathbb{N} \cup \{\infty\}$, and consider the F-algebra $R = \left(\begin{smallmatrix} F & K \\ 0 & F \end{smallmatrix}\right)$. Let A be the right ideal $\left(\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right)$ in R. For any ideal $I \subseteq_e R_R$, we have $I \cap A \neq 0$ and so $I \supseteq A$ since $\dim_F A = 1$. Therefore, $I \supseteq R \cdot A = \left(\begin{smallmatrix} 0 & K \\ 0 & F \end{smallmatrix}\right)$, so we have either $I = \left(\begin{smallmatrix} 0 & K \\ 0 & F \end{smallmatrix}\right)$, or $I = R$. From this, we see that $d(R) = 1$. Now let $\{u_i\}$ be an F-basis for K, and let $J_i = \left(\begin{smallmatrix} 0 & F \cdot u_i \\ 0 & 0 \end{smallmatrix}\right)$ in R. It is easy to check that the J_i's are ideals, and that $J := \sum_i J_i$ is a direct sum which is essential as an (R,R)-subbimodule in ${}_RR_R$. Since $\dim_F J_i = 1$ for each i, we see that $\text{u.dim}({}_RR_R) = n$. Therefore, $\text{u.dim}({}_RR_R)$ can be as far apart from $d(R)$ as one wants. In the case when $n < \infty$, it will be seen in (6.2) below that $Q := Q^{\mathrm{r}}_{\max}(R) \cong \mathbb{M}_{n+1}(F)$, so in particular $\text{u.dim}({}_QQ_Q) = 1$ (and $\text{u.dim}(R_R) = \text{u.dim}(Q_Q) = n + 1$). Therefore, we also have an example where the 2-sided uniform dimensions $\text{u.dim}({}_RR_R)$ and $\text{u.dim}({}_QQ_Q)$ differ by an arbitrary amount in (3.18) (3) (in the case $I = R$ and $S = Q$).

Note that in the above example, we have $d(R) = 1$ and yet $d(J) = n \leq \infty$ for the ideal $J = \left(\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right) \subseteq R$. The "trouble" here is that J is *not* right essential in R. Since J is 2-sided essential in ${}_RR_R$, this implies that $J \notin \mathscr{F}(R)$, which is the main source of the anomaly. See the remarks in the paragraph following (4.9) below.

4. Relating $d(I)$ to the direct product decompositions of $E(I)$

As we have pointed out in the Introduction, for a right nonsingular ring R, the maximal right ring of quotients $Q = Q^{\mathrm{r}}_{\max}(R)$ is a von Neumann regular right self-injective ring. The decomposition theory of such a ring Q into a (possibly infinite) direct product of prime rings is available from Goodearl's book [4]. In (9.11) of this book, it is shown that Q admits such a direct product decomposition iff $B(Q)$ is *atomic*, where $B(Q)$ denotes the (complete) Boolean algebra of central idempotents in Q. ("Atomic" here means that any nonzero $f \in B(Q)$ dominates some minimal element (atom) $f_0 \in B(Q)$.) In this section, we shall generalize our earlier result (3.3) by showing that, for any ideal $I \in \mathscr{F}(R)$, the study of arbitrary ideal direct sums $\bigoplus_{i \in C} I_i$ right essential in I_R corresponds exactly to the study of arbitrary direct product decompositions of the von Neumann regular right self-injective ring $E(I)$ associated with I. Using this correspondence, we can then deduce facts about the invariant $d(I)$ from known facts about the Boolean algebra $B(E(I))$ of central idempotents in $E(I)$. In the special case when $I = R$, for instance, this study recovers various known criteria for the maximal right quotient ring $Q^{\mathrm{r}}_{\max}(R)$ to be a prime ring: see (4.8).

Theorem 4.1. *Let R be a right nonsingular ring, and $Q=Q^{\mathrm{r}}_{\max}(R)$. Let $I\in\mathscr{F}(R)$, with $E(I)=eQ$ where $e\in B(Q)$. Let C be any (finite or infinite) indexing set.*

(1) If eQ is a direct product of rings $\prod_{i\in C}Q_i$, then $I_i:=Q_i\cap I$ $(i\in C)$ are ideals in R with $\bigoplus_{i\in C}I_i\subseteq_{\mathrm{e}}I_R$.

(2) If A_i $(i\in C)$ are mutually orthogonal right ideals in R such that $\bigoplus_{i\in C}A_i\subseteq_{\mathrm{e}}I_R$, then $Q_i:=E(A_i)$ $(i\in C)$ are rings with identity, with a ring isomorphism $eQ\cong\prod_{i\in C}Q_i$ (over R).

(3) For a given $i\in C$, assume that A_i in (2) is an ideal of R. Then Q_i is a prime ring iff $d(A_i)=1$.

(4) eQ is a direct product of prime rings iff there exist ideals $I_i\subseteq I$ with $d(I_i)=1$ for all i such that $\bigoplus_i I_i\subseteq_{\mathrm{e}}I_R$, iff every nonzero ideal $J\in\mathscr{B}(R)$ in I contains some $J_0\in\mathscr{B}(R)$ with $d(J_0)=1$.

Proof. (1) To begin with, note that $\bigoplus_i Q_i\subseteq_{\mathrm{e}}(eQ)_Q$. Since Q_R is nonsingular, it follows from (3.1)(2) that $\bigoplus_i Q_i\subseteq_{\mathrm{e}}(eQ)_R$. Now, from $I\subseteq_{\mathrm{e}}(eQ)_R$, we have $I_i=Q_i\cap I\subseteq_{\mathrm{e}}(Q_i)_R$. Therefore,

$$\bigoplus_i I_i\subseteq_{\mathrm{e}}\left(\bigoplus_i Q_i\right)_R\subseteq_{\mathrm{e}}(eQ)_R. \tag{4.2}$$

In particular, $\bigoplus_i I_i\subseteq_{\mathrm{e}}I_R$.

(2) Say $I\oplus J\subseteq_{\mathrm{e}}R_R$, where J is a suitable right ideal in R orthogonal to I. Then

$$A_i\oplus\left(J\oplus\bigoplus_{j\neq i}A_j\right)\subseteq_{\mathrm{e}}R_R, \tag{4.3}$$

with A_i orthogonal to $J\oplus\bigoplus_{j\neq i}A_j$. By (3.3) (in the case $n=2$), $E(A_i)$ is an ideal in Q of the form e_iQ, where $e_i\in B(Q)$. In particular, each $E(A_i)$ is a ring with identity e_i (necessarily in eQ). Since $\sum_i E(A_i)$ is a *direct* sum, the e_i's are mutually orthogonal. From $\bigoplus_i A_i\subseteq_{\mathrm{e}}I_R\subseteq_{\mathrm{e}}(eQ)_R$, we have $\bigoplus_i e_iQ\subseteq_{\mathrm{e}}(eQ)_R$ and a fortiori $\bigoplus_i e_iQ\subseteq_{\mathrm{e}}(eQ)_Q$. Since eQ is an injective Q-module, we have $E((\bigoplus_i e_iQ)_Q)=(eQ)_Q$. By [4, (9.9)], this means that $\bigwedge_{i\in C}e_i=e$ in the complete Boolean algebra $B(Q)$, and by [4, (9.10)] (applied to the von Neumann regular right self-injective ring eQ), this in turn implies that there is a ring isomorphism $eQ\cong\prod_{i\in C}e_iQ$ (given by $eq\mapsto(e_iq)_{i\in C}$ for any $q\in Q$).

(3) First assume $d(A_i)>1$. Then there exist nonzero ideals X, $Y\subseteq A_i$ such that $X\oplus Y\subseteq_{\mathrm{e}}(A_i)_R$. It follows from the above analysis that $E(X)$, $E(Y)$ are nonzero mutually orthogonal ideals in $Q_i=E(A_i)$, so Q_i is not a prime ring. Now assume that $d(A_i)=1$. We claim that Q_i is indecomposable as a ring. Indeed, if Q_i is a direct sum of two nonzero ideals X', Y', then, for the ideals $X=X'\cap A_i\neq 0$ and $Y=Y'\cap A_i\neq 0$, we have $X\oplus Y\subseteq_{\mathrm{e}}(A_i)_R$, which contradicts $d(A_i)=1$. Since the ring Q is von Neumann regular and right self-injective, so is eQ and each component ring Q_i. Having shown that Q_i is indecomposable, we conclude from [4, (9.6)] that Q_i is a prime ring.

(4) The first "iff" follows immediately from (1)–(3). The second "if" follows by taking Goodearl's "atomic" criterion for the decomposability of eQ into a direct product of prime rings, and translating it, via (3.15)(2), into a criterion in terms of the subideals of I in $\mathscr{B}(R)$. (We mention in passing that the second "iff" statement is also valid if we replace $\mathscr{B}(R)$ in both places by $\mathscr{F}(R)$.) $\square$

Remark 4.4. Note that, in the context of Theorem 4.1(4), the prime rings Q_i occurring in the direct product decomposition of eQ are in fact (left and right) primitive rings, by a result of Goodearl [2, Corollary 16]. Also, if eQ happens to be *left* self-injective, then each Q_i is left and right self-injective (and von Neumann regular), so each Q_i will in fact be a *simple* ring, by [4, (9.30)].

In the case of a *finite* indexing set C, we deduce easily the following result from Theorem 4.1.

Theorem 4.5. (*Notations as in* (4.1).) *For any natural number n, eQ is a direct product of n prime rings iff $d(I)=n$. (It follows, incidentally, that in this case $d(I)$ is exactly the number of atoms in the finite Boolean algebra $B(eQ)$.)*

In the case of an *infinite* indexing set C, a little additional work leads to the following (cf. (1.2)).

Theorem 4.6. (*Notations as in* (4.1).) *Suppose that $d(I)=\infty$. Then* (1) *there exist nonzero ideals $I_i \subseteq I$ $(i \geq 1)$ such that $\bigoplus_{i=1}^{\infty} I_i \subseteq_e I_R$, and* (2) *$eQ$ is an infinite direct product of nonzero rings.*

Proof. According to Theorem 4.1, (1) and (2) are equivalent statements, so it suffices to prove (2). Since $d(I)=\infty$, eQ cannot be a finite direct product of prime rings, so we must have $|B(eQ)|=\infty$. Write $e=e_1+e_1'$, where $0 \neq e_1, e_1' \in B(eQ)$. With a suitable labelling, we may assume that $|B(e_1'Q)|=\infty$. Next write $e_1'=e_2+e_2'$, with $0 \neq e_2, e_2' \in B(e_1'Q)$ and $|B(e_2'Q)|=\infty$, etc. In this way, we get an infinite set of nonzero mutually orthogonal central idempotents e_i's in eQ. Since $B(eQ)$ is a complete Boolean algebra, there exists a central idempotent $f := \bigvee_{i=1}^{\infty} e_i \in eQ$. Letting $e_0 := e - f \in B(eQ)$, we have then $\bigvee_{i=0}^{\infty} e_i = e$ where $\{e_0, e_1, e_2, \ldots\}$ are mutually orthogonal, and $e_i \neq 0$ for $i \geq 1$. By [4, (9.10)] again (applied to eQ), we have a ring isomorphism $eQ \cong \prod_{i=0}^{\infty} e_iQ$ (with $e_iQ \neq 0$ for $i \geq 1$), as desired. $\square$

Remark 4.7. There certainly exist right nonsingular rings R whose maximal right rings of quotients Q are *not* direct products of prime rings. We can construct a commutative example as follows. Let F be a field, and R be the commutative reduced ring $F \times F \times \cdots / M$ where $M = F \oplus F \oplus \cdots$. It is easy to check that R has no primitive idempotents, and hence that there is no ideal $J \subseteq R$ with $d(J)=1$. It thus follows from (4.1) that, for the (commutative) maximal quotient ring Q of R, there is no decomposition $Q \cong Q_1 \times Q_2$ where Q_1 is a prime ring (i.e. a field).

As a special case of (4.1) and (4.5), we can compile a list of characterizations (in terms of R) for the maximal right quotient ring Q to be a prime ring. In order to make all the statements directly accessible, we shall formulate them with only the annihilator notation (and not the more technical notations $\mathscr{F}(R)$ and $\mathscr{B}(R)$).

Theorem 4.8. *For R, Q as in (4.1) with $R \neq 0$, the following are equivalent:*
(1) *Q is a prime ring;*
(2) *For every ideal A in R, either $A'' = 0$ or $A'' = R$;*
(3) *For every ideal A in R, if $A = A''$, then either $A = 0$ or $A = R$;*
(4) *For every ideal A in R, if $A = A''$ and $A \cap A' = 0$, then either $A = 0$ or $A = R$;*
(5) *For every nonzero ideal A in R, $A' \neq 0$ implies $A \cap A' \neq 0$;*
(6) *If A, B are ideals in R such that $A \oplus B \subseteq_e R_R$, then either $A = 0$ or $B = 0$;*
(7) *If A, B are mutually orthogonal right ideals in R such that $A \oplus B \subseteq_e R_R$, then either $A = 0$ or $B = 0$.*

Proof. To avoid repetitions, we shall only give a sketch of the proof. Note that (6), (7) are just explicit statements for $d(R) = 1$. (4) is the statement that $\mathscr{B}(R) = \{0, R\}$ (see (3.11)), and (5) is the statement that every nonzero $A \in \mathscr{F}(R)$ is right essential in R. By our results in Sections 3 and 4, these are all equivalent to Q being a prime ring. The other conditions are technical variations of the ones mentioned above, and their equivalences can be checked readily. □

A few historical remarks about Theorem 4.8 are in order. The condition (4) in this theorem was discovered by Johnson in [8, p. 530]; he called a (nonzero) right nonsingular ring R *right irreducible* if R satisfies this condition. Later, Johnson introduced the equivalent condition (5) in [9, p. 712] (see also [11, p. 262]). For right nonsingular rings, Johnson proved that (4) (or (5)) implies (1) in [10, (2.7)], but it is not entirely clear that he proved the converse.[5] In [7], Handelman introduced the condition (2), and proved the equivalence of (1), (2), (4) as well as a couple of other conditions involving torsion and pretorsion theories. The equivalent condition (6) appeared in Theorem 6.1 of [3]. (3) and (7) do not seem to have appeared before, and are variations of the others.

We should also point out that the conditions for Q to be a simple ring (resp. a "full linear ring") were studied by Goodearl and Handelman in [5, (5.3)] (resp. [7, Corollary 8]), and the condition for Q to be a division ring is simply that R be a right Ore domain.

Remark 4.9. Suppose the right nonsingular ring R satisfies the strong finiteness condition $\text{u.dim}(R_R) < \infty$. By the theorem of Johnson and Gabriel (see, e.g. [16, p. 248]), this is precisely the case when Q is an (artinian) semisimple ring. In this case, Theorem 4.5 tells us that $d(R)$ computes the number of Wedderburn components of Q,

[5] In the literature, the full equivalence of (1) and (4) is sometimes attributed to Johnson (at least in the case when $\text{u.dim}(R_R) < \infty$); see, for instance, [14, p. 122].

and Theorem 4.8 gives a list of characterizations, in terms of R, for Q to be a simple artinian ring.

We close this section by making some remarks on the invariant $d(I)$. For a general (S,R)-bimodule I, the behavior of $d(I)$ seems rather mysterious. Firstly, if J is a subbimodule of I, we may not have $d(J) \leq d(I)$. In fact, as we have seen in the last paragraph of Section 3, it is possible for $d(I)$ to be 1 and $d(J)$ to be ∞. (This can be "corrected" by putting a condition on J: if $J \subseteq I \subseteq M$ are (S,R)-bimodules and $J \in \mathscr{F}(M)$, then it is easy to show that $d(J) \leq d(I)$.) Secondly, we do not know in general if $d(I) = \infty$ would imply that there are nonzero subbimodules $\{I_i \subseteq I\colon i \geq 1\}$ such that $\bigoplus_{i=1}^{\infty} I_i \subseteq_e I_R$. However, in (4.6), we were able to prove this property for $I \in \mathscr{F}(R)$ over a right nonsingular ring R. Similarly, by using the results in this section and by appealing to the known properties of $B(Q)$, we can derive a few other properties of the d-invariant for ideals in $\mathscr{F}(R)$ (over a right nonsingular ring R). We list below some of these properties (with only a sketch of their proofs).

(4.10) *For $\{I_i\colon 1 \leq i \leq n\} \subseteq \mathscr{F}(R)$, we have $d(\bigoplus_{i=1}^{n} I_i) = \sum_{i=1}^{n} d(I_i)$, with the usual conventions on the symbol ∞. (In particular, if each $d(I_i) = 1$, then $d(\bigoplus_{i=1}^{n} I_i) = n$.)*

(4.11) *For $I, J \in \mathscr{F}(R)$, we have $d(I) + d(J) = d(I+J) + d(I \cap J)$, with the usual conventions on ∞.*

Proof (*sketch*). For both cases, it is a simple matter of counting central idempotents in the injective hulls of the respective ideals (and using Theorem 4.1). For (4.11), we can reduce to the case of direct sums by using the familiar formula $eQ + fQ = eQ \oplus (1-e)fQ$ for idempotents. (Of course, it is also possible to prove (4.11) directly by using the analogous formula: $I \oplus (I' \cap J) \subseteq_e (I+J)_R$ for $I, J \in \mathscr{F}(R)$.) □

The property (4.11) for the d-invariant may be slightly surprising since the same formula is known to fail rather miserably for the usual one-sided uniform dimension of modules; see the paper of Camillo and Zelmanowitz [1].

5. Comparison of $d(I)$, u.dim$({}_RI_R)$ and u.dim(I_R)

For any (R,R)-bimodule I, the three invariants $d(I)$, u.dim$({}_RI_R)$ and u.dim(I_R) are in general related by

$$d(I) \leq \text{u.dim}({}_RI_R) \leq \text{u.dim}(I_R). \tag{5.1}$$

A natural question to ask is when are some of these invariants equal. We shall be primarily interested in the case when I is an ideal of R in the family $\mathscr{F}(R)$. For a general right nonsingular ring R, we have seen in (3.22) that, even for $I = R$, $d(I)$ and u.dim$({}_RI_R)$ can differ by any amount. So for equality to occur between two or all three of the invariants in (5.1), we have to look for special classes of R and I. Our first result in this direction is Theorem 5.3 below, which is preceded by the following lemma.

Lemma 5.2. *Let R be any ring, and I, J be ideals in R such that $J \cap J' = 0$. If $J \subseteq_e {}_R I_R$, then $J \subseteq_e I_R$.*

Proof. Let A be any right ideal in I such that $J \cap A = 0$. Then $A \cdot J \subseteq A \cap J = 0$, so $A \subseteq I \cap J'$. Now, since $J \subseteq_e {}_R I_R$ and J' is an ideal, $J \cap J' = 0$ implies that $I \cap J' = 0$. Therefore, $A = 0$, and this shows that $I \subseteq_e J_R$. □

Theorem 5.3. *Let R be any ring, and $I \subseteq R$ be any ideal which contains no nonzero nilpotent ideals of R. Then $d(I) = \text{u.dim}({}_R I_R)$.*

Proof. In view of (5.1), it suffices to prove that $\text{u.dim}({}_R I_R) \le d(I)$. Let $A := I_1 \oplus \cdots \oplus I_n$ be any direct sum of n nonzero ideals in I. Let I_0 be an ideal which is a 2-sided complement to A in I. (Such a complement always exists by Zorn's lemma.) Then $J := I_0 \oplus A \subseteq_e {}_R I_R$. Now $J \cap J'$ is an ideal of square zero in I, so by assumption, $J \cap J' = 0$. Therefore, by (5.2), we have

$$J = I_0 \oplus I_1 \oplus \cdots \oplus I_n \subseteq_e I_R.$$

This shows that $n \le d(I)$ (noting that I_0 is possibly zero), which then yields $\text{u.dim}({}_R I_R) \le d(I)$. □

Corollary 5.4. *Let R be any semiprime ring, For any ideal $I \subseteq R$, we have $d(I) = \text{u.dim}({}_R I_R)$. In the case when $I = R$, this is equal to the number $t\,(\le \infty)$ of minimal prime ideals in R.*

Proof. Since R contains no nonzero nilpotent ideals, the first conclusion follows from (5.3). The fact that $\text{u.dim}({}_R R_R) = t$ is proved in [15, p. 54]. □

Remarks 5.5. (1) The number $\text{u.dim}({}_R R_R) = t$ is called the "prime dimension" of R by Kharchenko [12]. If this number t is finite, then, as Kharchenko pointed out in [12, p. 167], each of the left, right and symmetric Martindale rings of quotients of R is a direct product of t prime rings.

(2) In the case when R is a semiprime right Goldie ring, one has $Q^{\text{r}}_{\max}(R) = Q^{\text{r}}_{\text{cl}}(R)$, the classical right ring of quotients of R. In this case, (5.4) and (4.9) together imply that the number of Wedderburn components of $Q^{\text{r}}_{\text{cl}}(R)$ is given by the number t of minimal prime ideals in R. This is a well known fact; see [15, 3.2.2, p. 68].

(3) In the case when R is a right nonsingular ring, its maximal right quotient ring Q is certainly semiprime. Therefore, (3.18) and (5.4) together imply that $d(R)$ is equal to $d(Q) = \text{u.dim}({}_Q Q_Q)$, and hence also equal to the number t $(\le \infty)$ of minimal prime ideals of Q.

(4) From (5.4), we can also quickly recover Johnson's result [9, (2.1)] that a semiprime right irreducible ring is always prime.

Theorem 5.6. *Let R be any right nonsingular ring such that its symmetric maximal quotient ring $Q_\sigma(R)$ is semiprime. Then, for any $I \in \mathscr{F}(R)$, we have $d(I) =$ u.dim$({}_R I_R)$.*

Proof. Let $S = Q_\sigma(R)$ and $I^* = SIS \subseteq S$. We have, by (3.18) (4), u.dim$({}_R I_R) =$ u.dim$({}_S I_S^*)$. By assumption, S is semiprime, so by (5.4) (applied to S), u.dim$({}_S I_S^*) = d(I^*)$ (where $d(I^*)$ means $d({}_S I_S^*)$). Finally, by (3.18) (1), $d(I^*) = d(I)$. Combining these equations, we have then u.dim$({}_R I_R) = d(I)$. □

To name some classes of right nonsingular rings to which (5.6) can be applied, recall that a right nonsingular ring R is said to be *right Utumi* if $A' \neq 0$ for any nonessential right ideal $A \subseteq R$ (see [16, p. 251] for other equivalent definitions). A left Utumi ring is defined similarly. By a Utumi ring, we shall always mean a right and left Utumi ring. A basic result in the theory of maximal quotient rings, due to Utumi, states that, *for any right nonsingular ring R, $Q^r_{max}(R) = Q^\ell_{max}(R)$ iff R is a Utumi ring*; see [3, (2.38)], or [16, (4.9), p. 252]. For such a ring R, we can deduce from (5.6) that the first two invariants in (5.1) are always equal for ideals in $\mathscr{F}(R)$.

Corollary 5.7. *Let R be a Utumi ring. Then, for any $I \in \mathscr{F}(R)$, $d(I) =$ u.dim$({}_R I_R)$.*

Proof. By the Utumi assumption on R, we have $Q := Q^r_{max}(R) = Q^\ell_{max}(R)$. In particular, $Q = Q_\sigma(R)$. Since Q is von Neumann regular, it is semiprime. Therefore, (5.6) applies. □

Combining (5.7) with (4.9), we have the following:

Corollary 5.8. *For any Utumi ring R:*

(1) R is right irreducible iff it is left irreducible, and in this case any two nonzero ideals in R intersect nontrivially.

(2) If u.dim$(R_R) < \infty$, *then the number of Wedderburn components of the semisimple ring Q is given by* u.dim$({}_R R_R)$.

Of course, in (5.3), (5.4) and (5.7), $d(I) =$ u.dim$({}_R I_R)$ may not be equal to u.dim(I_R) in general, as the classical cases of non-Ore domains and non-reduced semisimple rings already show. Let us now investigate some circumstances in which u.dim(I_R) can be equal to $d(I)$ or u.dim$({}_R I_R)$. Crucial to this consideration is the following condition on the ideal I:

(*) *For any right ideals $A, A' \subseteq I$, $A \cap A' = 0 \Rightarrow A \cdot A' = 0$.*

Note that this condition fails to hold for $I = R$ over a non-Ore domain, and also over a non-reduced semisimple ring. For ideals I satisfying the condition (*), we have the following positive result.

Theorem 5.9. *Let R be a ring, and let $I \subseteq R$ be any ideal satisfying the condition (*). Assume that either*

(1) *I contains no nonzero nilpotent ideal of R, or*

(2) *R is right nonsingular and $I \in \mathscr{F}(R)$.*

Then $d(I) = \text{u.dim}(_R I_R) = \text{u.dim}(I_R)$.

Proof. (1) We know already from (5.3) that, in this case, $d(I) = \text{u.dim}(_R I_R)$, so it only remains to show that $\text{u.dim}(I_R) \le \text{u.dim}(_R I_R)$. Consider any direct sum of nonzero right ideals $A_1 \oplus \cdots \oplus A_n \subseteq I$, and let $B_i := R \cdot A_i \subseteq I$ $(1 \le i \le n)$ be the ideals generated by A_i. We are done if we can show that the sum of ideals $\sum_{i=1}^n B_i \subseteq I$ is *direct*. For ease of notations, let us just show that the intersection $C := B_1 \cap (B_2 + \cdots + B_n)$ is zero. By (*), $A_i \cdot A_j = 0$ for any $i \ne j$. Therefore, for $i \ne j$,

$$B_i \cdot B_j = (RA_i)(RA_j) = R(A_i R)A_j = R \cdot A_i A_j = 0,$$

and hence $C^2 = C \cdot C \subseteq B_1 \cdot (B_2 + \cdots + B_n) = 0$. Since $C \subseteq I$, our assumption on I in this case yields $C = 0$, as desired.

(2) In view of (5.1), we need only show that $\text{u.dim}(I_R) \le d(I)$. Consider any direct sum of nonzero right ideals $A := A_1 \oplus \cdots \oplus A_n \subseteq I$. Let A_0 be a right ideal which is a complement to A in I_R. Then $A_0 \oplus A = \bigoplus_{i=0}^n A_i \subseteq_e I_R$. By the condition (*) on I, the right ideals $A_i \subseteq I$ $(0 \le i \le n)$ are mutually orthogonal. Since R is right nonsingular and $I \in \mathscr{F}(R)$, (3.16) implies that $n \le d(I)$ (noting that A_0 is possibly zero). This shows that $\text{u.dim}(I_R) \le d(I)$, as desired. □

Corollary 5.10. *Let R be a reduced right Utumi ring. Then for any ideal $I \subseteq R$, we have* $d(I) = \text{u.dim}(_R I_R) = \text{u.dim}(I_R)$.

Proof. The fact that R is reduced right Utumi implies that any ideal I satisfies (*) (by [16, (5.2), p. 254]), and (of course) that R has no nonzero nilpotent ideals. Therefore, the desired conclusion follows from Case (1) of (5.9). □

Remark 5.11. (1) By symmetry, it follows from (5.10) that, if R is any reduced Utumi ring R, then $\text{u.dim}(_R I) = \text{u.dim}(I_R)$ for any ideal I.

(2) The reducedness property used in the proof of (5.10) is only a sufficient, but not a necessary, condition. In fact, the conclusion in (5.10) clearly holds for any ideal I in any commutative ring R. More generally, if R is any *right duo* ring (i.e., a ring in which any right ideal is an ideal), it is easy to check that all ideals I belong to $\mathscr{F}(R)$ and also satisfy (*), and that the conclusion of (5.10) holds for I.

We close this section with one more result on the comparison between the three invariants in (5.1). This result requires no assumptions whatsoever on the ring R.

Proposition 5.12. *For any ring R and an ideal $I \subseteq R$, suppose that $n := \text{u.dim}({}_R I_R) = \text{u.dim}(I_R) < \infty$. Then $d(I) = n$ too.*

Proof. Let $I_1, \ldots, I_n$ be nonzero ideals such that $I_1 \oplus \cdots \oplus I_n \subseteq_e RI_R$. Since $\text{u.dim}(I_R) = n$, we must have already $I_1 \oplus \cdots \oplus I_n \subseteq_e I_R$. Therefore, $n \leq d(I)$, from which we conclude that $d(I) = n$. □

Remark 5.13. In contrast to (5.12), in case $\text{u.dim}({}_R I_R) = \text{u.dim}(I_R) = \infty$, we may *not* have $d(I) = \infty$. We have already constructed such an example in (3.22), with $I = R$. In fact, in the notation of (3.22), if $\dim_F K = \infty$, then $\text{u.dim}({}_R R_R) = \text{u.dim}(R_R) = \infty$, but $d(R) = 1$! The same example shows that no conclusions can be drawn on $d(I)$ if $\text{u.dim}({}_R I_R) = n$ and $\text{u.dim}(I_R) = n + 1$ for some finite n.

6. Examples

We conclude with a few illustrative examples in this section. In (5.7), we have shown that the Utumi condition $Q^{r}_{\max}(R) = Q^{\ell}_{\max}(R)$ implies $d(I) = \text{u.dim}({}_R I_R)$ for any ideal $I \in \mathscr{F}(R)$. The converse is not true. For instance, take a domain R which is right Ore but not left Ore. Then $d(I) = \text{u.dim}({}_R I_R) = \text{u.dim}(I_R) = 1$ for any nonzero ideal I. Here, $Q^{r}_{\max}(R)$ is a division ring, and $Q^{\ell}_{\max}(R)$ is not even a reduced ring. The ring R is easily checked to be right Utumi but not left Utumi. In the following, we shall construct, for *any* given $n \geq 2$, an example of a (right nonsingular) ring R for which $d(R) = \text{u.dim}({}_R R_R) = n - 1$, but $Q^{r}_{\max}(R)$ is *not* isomorphic to $Q^{\ell}_{\max}(R)$.

Example 6.1. Let F be any field, and $n \geq 2$ be a natural number. Let R be the ring $\sum_{i=1}^{n} F e_{ii} + \sum_{i=2}^{n} F e_{1i}$, where the e_{ij}'s are matrix units in $\mathbb{M}_n(F)$. To compute $Q := Q^{r}_{\max}(R)$, let $P := (\mathbb{M}_2(F))^{n-1}$ (direct product of $n-1$ copies of $\mathbb{M}_2(F)$), and consider the ring embedding $\varphi : R \to P$ defined by

$$\varphi\left(\sum_{i=1}^{n} a_{ii}e_{ii} + \sum_{i=2}^{n} a_{1i}e_{1i}\right) = (a_{11}e_{11} + a_{ii}e_{22} + a_{1i}e_{12})_{i \geq 2} \in P.$$

It is easy to check that $\varphi(R) \subseteq_e P_R$ and that P is von Neumann regular and self-injective, so it follows from [3, (2.11)] that R is right nonsingular, with $Q \cong P$. Therefore, $d(R) = d(Q) = n - 1$ by (3.18) (1) and (3.21). Now consider

$$I_i = Fe_{1i} + Fe_{ii} \subseteq R \quad (i \geq 2).$$

It is easy to check that these are minimal ideals in R, with $\bigoplus_{i=2}^{n} I_i \subseteq_e R_R$. From this, we see that u.dim$({}_RR_R)=n-1=d(R)$. However, since $R \supseteq \sum_{i=1}^{n} Fe_{1i}$, $Q_{\max}^{\ell}(R)$ is given by $\mathbb{M}_n(F)$, which is *not* isomorphic to $Q_{\max}^{r}(R)$ as long as $n>2$. (In particular, R is not Utumi if $n>2$.)

It is worth pointing out that the constructions in (6.1) can actually be extended to the case of infinite matrices. To see this, let $R=\sum_{i=1}^{\infty} Fe_{ii}+\sum_{i=2}^{\infty} Fe_{1i}$ instead. Then similar arguments can be used to show that $Q_{\max}^{r}(R)\cong \mathbb{M}_2(F)^{\infty}$ (countable infinite direct product of copies of $\mathbb{M}_2(F)$), and $Q_{\max}^{\ell}(R)\cong \mathbb{M}_{\infty}^{\mathrm{cf}}(F)$ (the ring of column-finite infinite matrices over F). We have here $d(R)=$ u.dim$({}_RR_R)=\infty$, and $d'(R)=1$, where $d'(R)$ denotes the d-invariant for the bimodule ${}_RR_R$ defined by giving preference to *left* (instead of right) essentialness in R.

We stress that the Utumi condition $Q_{\max}^{r}(R)=Q_{\max}^{\ell}(R)$ in (5.7) amounts to the fact that the two quotient rings are not just isomorphic, but are isomorphic over R. To illustrate this point, we'll construct below an example where $Q_{\max}^{r}(R)$ and $Q_{\max}^{\ell}(R)$ are isomorphic as rings, but nevertheless $d(R)\neq$ u.dim$({}_RR_R)$.

Example 6.2. Let R be the F-algebra constructed in (3.22). We will use the notations in (3.22), but assume here that $n=\dim_F K$ is finite and greater than 1. For a fixed F-basis $\{u_1,\ldots,u_n\}$ on K, consider the F-algebra embeddings

$$\varphi_1,\ \varphi_2 : R \to \mathbb{M}_{n+1}(F)$$

defined by sending the matrix

$$x=\begin{pmatrix} a & \sum_i c_i u_i \\ 0 & b \end{pmatrix} \in R$$

to, respectively,

$$\varphi_1(x)=\begin{pmatrix} a & c_1 \cdots c_n \\ 0 & \\ \vdots & b\cdot I_n \\ 0 & \end{pmatrix} \quad \text{and} \quad \varphi_2(x)=\begin{pmatrix} & c_1 \\ a\cdot I_n & \vdots \\ & c_n \\ 0 \cdots 0 & b \end{pmatrix},$$

where $a,b,c_1,\ldots,c_n\in F$. One can check that $\varphi_1(R)$ (resp. $\varphi_2(R)$) is left (resp. right) essential in $\mathbb{M}_{n+1}(F)$ as a $\varphi_1(R)$-module (resp. $\varphi_2(R)$-module). Thus,

$$Q_{\max}^{\ell}(\varphi_1(R))=\mathbb{M}_{n+1}(F)=Q_{\max}^{r}(\varphi_2(R)),$$

again by [3, (2.11)]. In particular, we have $Q_{\max}^{\ell}(R)\cong Q_{\max}^{r}(R)\cong \mathbb{M}_{n+1}(F)$ as rings. However, as we have shown in (3.22), u.dim$({}_RR_R)=n>1=d(R)$. This implies that the two quotient rings *cannot* be isomorphic over R, which one can also check directly. In fact, the nonsingular ring R here is neither left nor right Utumi. Moreover, the symmetric maximal ring of quotients $Q_\sigma(R)$ turns out to be the ring R itself, so (3.18) (4) does not yield any useful information about R.

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