# RINGS HAVING SOLVABLE ADJOINT GROUPS

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ABSTRACT. Let $^\circ R$ denote the group of quasi-regular elements of a ring $R$ with respect to circle operation. The following results have been proved: (1) If $R$ is a perfect ring and $^\circ R$ is finitely generated solvable group then $R$ is finite and hence $^\circ R = P_1 \circ P_2 \circ \cdots \circ P_m$ where $P_i$ are pairwise commuting $p$-groups. (2) Let $R$ be a locally matrix ring or a prime ring with nonzero socle. Then $^\circ R$ is solvable iff $R$ is either a field or a $2\times 2$ matrix ring over a field having at most 3 elements.

For a ring $R$ let $J(R)$ denote the Jacobson radical, $^\circ R$ the group of quasi-regular elements with respect to circle operation and $U(R)$ the group of units if $R$ has identity. We know that if $R$ has identity, then $^\circ R$ is isomorphic to $U(R)$. $^\circ R$ is called the adjoint group of $R$. The object of this paper is to study certain classes of rings $R$ for which $^\circ R$ is nilpotent, supersolvable or solvable.

1.1. Let $M$ be a unital free $R$-module over a ring $R$ in which 2 is invertible and let $U(S)$ be supersolvable where $S = \mathrm{Hom}_R(M, M)$. Then $S = R$.

If $S \neq R$, then $S$ contains a copy $T$ of a $2\times 2$ matrix ring over $R$. In this case we choose $a, b, c \in U(T)$ such that

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}, \qquad c = \begin{pmatrix} -1 & \frac{1}{2} \\ -2 & 1 \end{pmatrix}.$$

Direct computation yields $a^{-1}b^{-1}ab = a$, $b^{-1}c^{-1}bc = b$. These relations first imply that $a$, $b$ belong to the derived group of $U(T)$ and further the relation $a^{-1}b^{-1}ab = a$ implies that the derived group cannot be nilpotent. Hence $U(T)$ cannot be supersolvable in contradiction to the hypothesis that $U(S)$ is supersolvable. This proves 1.1.

The following example shows that if $2 = 0$ in a ring $R$ then 1.1 may not be true: The group of units of a $2\times 2$ matrix ring over a field of 2 elements is supersolvable.

1.2. Let $M$ be a unital free module over a ring $R$ of characteristic 2 and let the group $U(S)$ be nilpotent where $S = \mathrm{Hom}_R(M, M)$. Then $S = R$.

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We argue as in 1.1. Here we choose

$$a=\begin{pmatrix}0&1\\1&1\end{pmatrix},\qquad b=\begin{pmatrix}1&0\\1&1\end{pmatrix}.$$

Again $a^{-1}b^{-1}ab=a$ and this shows that the subgroup consisting of all the $2\times 2$ matrices over $R$ cannot be nilpotent, in contradiction to the hypothesis that the group $U(S)$ is nilpotent. Hence $S=R$.

1.3. (Hua). If $D$ is a division ring and $^\circ D$ is solvable then $D$ is a field.

1.4. The group of units of the ring of all linear transformations of a vector space $V$ over a division ring $D$ is nilpotent iff $\dim_D V=1$ and $D$ is a field.

Since nilpotent groups are solvable the result follows from 1.2 and 1.3.

1.5. (Scott). Let $F$ be a field, $G=\mathrm{SL}(2,\ F)$ and $\#(F)>3$. Then $G=G^1$.

1.6. (Dickson). The group of quasi-regular elements in the $3\times 3$ matrix ring over a field having at most 3 elements is not solvable.

1.7. The group of units of the ring of all linear transformations of a vector space $V$ over a division ring $D$ is solvable iff either $D$ is any field and $\dim_D V=1$ or $D$ is a field having at most 3 elements and $\dim_D V=2$.

This is a consequence of 1.3, 1.5 and 1.6.

1.8. If $R$ is any ring such that $R| J(R)$ is artinian and $^\circ R$ is solvable then $R| J(R)$ is a finite direct sum of rings $R_i$ where $R_i$ is a field or a $2\times 2$ matrix ring over a field having at most 3 elements.

Since homomorphic image of a solvable group is solvable, the result follows from 1.7.

1.9. (Bass). $R$ is a ring with dcc for principal right ideals iff $R| J(R)$ is artinian and $J(R)$ is $T$-nilpotent.

Such rings have been called perfect rings by Bass.

1.10. If $R$ is a perfect ring and $^\circ R$ is a finitely generated solvable group then $R$ is finite, and hence $^\circ R=P_1\circ\cdots\circ P_m$ where $P_i$ are pairwise commuting $p$-groups.

Since the fields whose multiplicative groups are finitely generated are Galois fields, finiteness of $R| J(R)$ follows at once from 1.8 and 1.9. This implies $^\circ J(R)$ is also finitely generated. Further $J(R)$ is locally nilpotent since it is $T$-nilpotent. Hence $J(R)$ is nilpotent which implies $^\circ J(R)$ is nilpotent and therefore by Watters $J(R)^+$ is finitely generated. Since $R$ is perfect, we get $J(R)$ is finite. Hence $R$ is finite. The last assertion is a consequence of Hall's well-known theorem for finite solvable groups.

2. We now proceed to characterise the class of prime rings, with nonzero socles, for which $^\circ R$ is solvable.

We denote by $C$ the class of all rings which are either fields or $2\times 2$ matrix rings over fields having at most 3 elements. In what follows it is assumed that $^\circ R$ is solvable.

2.1. A locally matrix ring $R$ over a division ring is in $C$.

We recall that $R$ is a locally matrix ring over a division ring $\Delta$ if each finite subset is in a subring of $R$ isomorphic to an $n\times n$ matrix ring over $\Delta$, for some natural number $n$. If $R$ is not a $2\times 2$ matrix ring over a field having at most 3 elements or it is not a field having less than 82 elements then $R$ must contain at least 82 distinct elements $a_i$. These $a_i$ can be imbedded in a subring $S$ of $R$ where $S$ is an $n\times n$ matrix ring over $\Delta$. Since $^\circ S$ is a subgroup of $^\circ R$, the subgroup $^\circ S$ is also solvable. By 1.7 we get that $S\cong\Delta$ is a commutative field. Since $S$ contains arbitrary elements of $R$, we get $R$ is commutative and hence $R$ is a field proving that $R$ is in $C$.

Since by Litoff theorem simple ring with minimal one-sided ideals is a locally matrix ring over a division ring 2.1 gives

COROLLARY. *Simple rings with minimal one-sided ideals are in the class $C$.*

2.2. If $R$ is a prime ring with nonzero socle then $R$ is in $C$. We know that $R$ is then a primitive ring with nonzero socle.

The socle is a simple ring with minimal one-sided ideals. Thus by 2.1 socle (as a ring) has identity. But then the identity of the socle (which is also an ideal in $R$) is easily shown to be the identity for the whole ring. Hence $R=$ socle, proving that $R$ is in the class $C$.

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