Nonnegative Matrices Having Nonnegative Drazin Pseudoinverses

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ABSTRACT

Necessary and sufficient conditions for nonnegative matrices having nonnegative Drazin pseudoinverses are obtained. A decomposition theorem which characterizes the class of all nonnegative matrices with nonnegative Drazin pseudoinverses is proved, thus answering a question raised by several people. It is also shown that if a row (or column) stochastic matrix has a nonnegative Drazin pseudoinverse $A^{(d)}$, then $A^{(d)}$ is some power of A. These results extend known results for nonnegative group-monotone matrices.

1. PRELIMINARIES AND DEFINITIONS

For every square matrix A (real or complex) there is a unique solution X to the equations $AX=XA$, $A^k=A^{k+1}X$ for some positive integer k, and $X=X^2A$. The unique solution X is called the Drazin pseudoinverse (or simply Drazin inverse) of A and is written as $A^{(d)}$. The smallest positive integer k such that $A^k=A^{k+1}X$ is called the (Drazin) index of A. If $k=1$, the Drazin inverse of A is called the group inverse of A and is denoted by $A^{\#}$.

If N is a matrix such that $N^{k-1}\neq 0$, $N^k=0$, then k is called the (nilpotency) index of N.

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A matrix $A=(a_{ij})$ is called nonnegative if $a_{ij}\geqslant 0$, and this is expressed as $A\geqslant 0$. An $n\times n$ square matrix $A=(a_{ij})$ is called row (or column) stochastic if $a_{ij}\geqslant 0$ and $\sum_{j=1}^{n}a_{ij}=1$ (or $a_{ij}\geqslant 0$ and $\sum_{i=1}^{n}a_{ij}=1$). A matrix is called doubly stochastic if it is both row and column stochastic. If a matrix A is a direct sum of matrices S_i, then the matrices S_i are called summands of A.

Let A be any square matrix, and let k be the index of A. It is shown in [3] that either A is nilpotent or

$$A^k=\prod_{i=1}^{k}F_i\prod_{i=1}^{k}G_{k+1-i},$$

where each of the matrices $F_1,\dots,F_k$ and $\prod_{i=1}^{k}F_i$ has full column rank and each of the matrices $G_1,\dots,G_k$ and $\prod_{i=1}^{k}G_{k+1-i}$ has full row rank. Also, the following relations hold:

$$A=F_1G_1$$

and

$$G_iF_i=F_{i+1}G_{i+1},\qquad i=1,2,\dots,k-1;$$

$(G_kF_k)^{-1}$ exists, and

$$A^{(d)}=\prod_{i=1}^{k}F_i(G_kF_k)^{-k-1}\prod_{i=1}^{k}G_{k+1-i}.$$

The above rank factorizations of powers of a matrix will be called the Cline's rank factorizations. In addition, if $\prod_{i=1}^{k}F_i$ and $\prod_{i=1}^{k}G_{k+1-i}$ are nonnegative, then the matrix A^k is said to have nonnegative rank factorization. We give, in Theorem 2, a necessary and sufficient condition for a nonnegative matrix to have a nonnegative Drazin pseudoinverse via Cline's rank factorization.

The proof of our Theorem 3, which characterizes the class of all nonnegative matrices having nonnegative Drazin pseudoinverses, makes use of the following result proved in [8].

THEOREM A [8, Theorem 1]. *Let A be a nonnegative matrix and $A^{\#}=p(A)\geqslant 0$, where $p(A)=\alpha_1A^{m_1}+\cdots+\alpha_sA^{m_s}$, $\alpha_i\neq 0$, $m_i>0$. Then there exists a permutation matrix P such that*

$$PAP^T=\begin{bmatrix} J & JD & 0 & 0\\ 0 & 0 & 0 & 0\\ CJ & CJD & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

where C, D are some nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

(I) $\beta x y^T$, *where x, y are positive vectors with $y^T x = 1$ and β is a positive root of* $\sum_{i=1}^{s} \alpha_i t^{m_i+1} = 1$,

(II)

$$\begin{bmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1,d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where x_i and y_i are positive vectors of the same order with $y_i^T x_i = 1$, x_i and x_j, $i \neq j$, are not necessarily of the same order, $d | m_{i+1}$ for some m_i, $d > 1$, and $\beta_{12}, \ldots, \beta_{d1}$ are arbitrary positive numbers such that their product $\beta_{12}\beta_{23} \cdots \beta_{d1}$ is a common root of the following system of at most d equations in t:

$$\sum_{d | m_i + 1} \alpha_i t^{(m_i+1)/d} = 1,$$

$$\sum_{d | m_i + 1 - k} \alpha_i t^{(m_i+1-k)/d} = 0, \qquad k \in \{1, \ldots, d-1\},$$

where the summation in each of the above equations runs over all those m_i for which $d | m_i + 1 - k, k = 0, 1, \ldots, d-1$, with the convention that if there is no m_i for which $d | m_i + 1 - k$, $k \in \{1, \ldots, d-1\}$, then the corresponding equation is absent.

Conversely, if for some permutation matrix P

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where C, D are arbitrary nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

(I′) βxy^T, $\beta > 0$, x, y *are positive vectors with* $y^Tx = 1$,
(II′)

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1,d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\beta_{ij} > 0$ x_i *and* y_i *are positive vectors with* $y_i^Tx_i = 1$, *then* $A^{\#}$ *exists and is nonnegative.*

Throughout this paper the matrices of types (I), (II), (I′), or (II′) will mean the types described in Theorem A.

2. NECESSARY AND SUFFICIENT CONDITIONS

In the following, for any matrix A, $N(A)$ and $R(A)$ will denote respectively the null space and the range of A. Also, for any positive integer n, R_+^n will denote the nonnegative orthant of the n-dimensional Euclidean space. Theorem 1 extends the theorem of Berman and Plemmons [1] to matrices of index k, and the proof is exactly on the same lines as that of Berman and Plemmons.

THEOREM 1. *Let A be an $n\times n$ nonnegative matrix of index k. Then $A^{(d)} \geqslant 0$ if and only if*

$$Ax \in R_+^n + N(A^k), \quad x \in R(A^k) \quad \Rightarrow \quad x \geqslant 0.$$

Proof. Let $A^{(d)} \geqslant 0$, and $Ax \in R_+^n + N(A^k)$, $x \in R(A^k)$. Write $Ax = x_1 + n$, $x_1 \in R_+^n$, and $n \in N(A^k) = N(A^{(d)})$. Now $x \in R(A^k)$ implies $x = A^kb$ for some vector b. So $x = A^kb = A^{k+1}A^{(d)}b = A^{(d)}A(A^kb) = A^{(d)}Ax = A^{(d)}(x_1 + n) = A^{(d)}x_1 \geqslant 0$, as desired.

Conversely, in order to prove that $A^{(d)} \geqslant 0$, we shall prove that for each nonnegative vector $x, A^{(d)}x \geqslant 0$. Since A is of index k, we can write $x = x_1 + x_2$ where $x_1 \in R(A^k)$, $x_2 \in N(A^k) = N(A^{(d)})$. Then $A^{(d)}x = A^{(d)}x_1 + A^{(d)}x_2 = A^{(d)}x_1$. Since $x_1 \in R(A^k)$, $AA^{(d)}x_1 = x_1 = x - x_2 \in R_+^n + N(A^k)$. Thus by hypothesis $A^{(d)}x_1 \geqslant 0$. Hence $A^{(d)}x = A^{(d)}x_1$ yields $A^{(d)}x \geqslant 0$ as desired. ∎

THEOREM 2. *Let A be a nonnegative matrix of index k. If $A^k = \prod_{i=1}^k F_i \prod_{i=1}^k G_{k+1-i}$, where each of the matrices $F_1,\ldots,F_k$ has full column* *rank with* $\prod_{i=1}^{k} F_i \geq 0$, *and each of the matrices* $G_1, \dots, G_k$ *has full row rank, with* $\prod_{i=1}^{k} G_{k+1-i} \geq 0$, *such that* $G_i Fi = F_{i+1} G_{i+1}$, $i = 1, 2, \dots, k-1$. *Then* $A^{(d)} \geq 0$ *if and only if* $(G_k F_k)^{-1-k} \geq 0$.

Proof. Let $(G_k F_k)^{-1-k} \geq 0$. From [3],

$$A^{(d)} = \prod_{i=1}^{k} F_i (G_k F_k)^{-1-k} \prod_{i=1}^{k} G_{k+1-i}.$$

Since $(G_k F_k)^{-1-k} \geq 0$, we get $A^{(d)} \geq 0$.

Conversely, let $A^{(d)} \geq 0$. Since $A^k = (\prod_{i=1}^{k} F_i)(\prod_{i=1}^{k} G_{k+1-i})$ is a nonnegative rank factorization of A^k and $(A^k)^{\#} = (A^{(d)})^k \geq 0$ by [2, Lemma 2], the left inverse $(\prod_{i=1}^{k} F_i)_L$ and right inverse $(\prod_{i=1}^{k} G_{k+1-i})_R$ are both nonnegative. From [3], $(G_k F_k)^{-1-k} = (\prod_{i=1}^{k} F_i)_L A^{(d)} (\prod_{i=1}^{k} G_{k+1-i})_R$. Thus $(G_k F_k)^{-1-k} \geq 0$, completing the proof. ■

REMARK 1. Whether every nonnegative matrix A with $A^{(d)} \geq 0$ has Cline's rank factorization such that $\prod_{i=1}^{k} F_i$ and $\prod_{i=1}^{k} G_{k+1-i}$ are nonnegative is open.

REMARK 2. For a given Cline's rank factorization of any square matrix A (not necessarily nonnegative) of index k, it can be shown that

$$A^{(d)} = A^k p(A)$$

if and only if

$$(G_k F_k)^{-1} = (G_k F_k)^k p(G_k F_k),$$

where $p(x)$ is some polynomial in x with scalar coefficients. The above interesting result follows by straightforward computations.

3. DECOMPOSITION THEOREM

We observe that the class of nonnegative matrices having nonnegative Drazin pseudoinverse contains matrices of the form

$$\begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & CKD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Where C, D are arbitrary nonnegative matrices and

$$K = \begin{pmatrix} J & 0 \\ 0 & N \end{pmatrix},$$

in which J is a direct sum of matrices of the types (I′) and (II′) (not necessarily both) and

$$N=\begin{bmatrix} 0 & C_{12} & C_{13} & \cdots & C_{1l} \\ 0 & 0 & C_{23} & \cdots & C_{2l} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & C_{l-1,l} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Where 0's on the diagonal stand for square matrices and the C_{ij}'s are nonnegative matrices of appropriate orders. If $N\neq 0$, then the above class is strictly larger than the class of all nonnegative matrices having nonnegative group inverses (Theorem A). The following example shows that not every nonnegative matrix A with $A^{(d)}\geqslant 0$ is in the above class. Consider

$$A=\begin{bmatrix} a+\frac{1}{3} & -a+\frac{1}{3} & \frac{1}{3} \\ a+\frac{1}{3} & -a+\frac{1}{3} & \frac{1}{3} \\ -2a+\frac{1}{3} & 2a+\frac{1}{3} & \frac{1}{3} \end{bmatrix}, \qquad -\tfrac{1}{6}\leqslant a\leqslant \tfrac{1}{6},\quad a\neq 0.$$

Here $A=xx^T+N$, $xx^TN=0=Nxx^T$, N is nilpotent, and

$$N=\begin{bmatrix} a & -a & 0 \\ a & -a & 0 \\ -2a & 2a & 0 \end{bmatrix}, \qquad x=\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Also, $A^2=xx^T$, $A^2=A^3$, $A^{(d)}=A^2$, and index$A=2$. If A were in the class described above, then this being stochastic $K=J$ (N must be absent) and thus by Theorem A, the matrix A is of index 1, a contradiction.

Recall that the Drazin pseudoinverse of a matrix A of index k is a polynomial $p(A)$ in A with scalar coefficients and is of the form

$$p(A)=\sum_{i=1}^{s}\alpha_1 A^{m_i}, \qquad m_i\geqslant k.$$

THEOREM 3. *Let $A\geqslant 0$ and $A^{(d)}\geqslant 0$. Let $A^{(d)}=p(A)=\sum_{i=1}^{s}\alpha_i A^{m_i}$, $\alpha_i\neq 0$, $m_i\geqslant k$. Then*

$$A=A_1+\cdots+A_r+N,$$

where

$$A_i \geqslant 0, \qquad A_iA_j = 0, \quad i \neq j,$$
$$A_iN = 0 = NA_i, \qquad 1 \leqslant i, j \leqslant r,$$

N is a nilpotent matrix of the (nilpotency) index k, and for some permutation matrices P_i,

$$P_iA_iP_i^T = \begin{bmatrix} G & GD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CG & CGD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in which C, D are nonnegative matrices of appropriate sizes, and G is a matrix of type (I) or (II).

Conversely, suppose A is a nonnegative matrix of index k; $A = A_1 + \cdots + A_r + N$; $A_i > 0$; $A_iA_j = 0$, $i \neq j$; $A_iN = 0 = NA_i$, $1 < i, j < r$, *N is nilpotent of the (nilpotency) index k; and for some permutation matrices* P_i

$$P_iA_iP_i^T = \begin{bmatrix} G & GD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CG & CGD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where G is a matrix of type (I′) or (II′) and C, D are arbitrary nonnegative matrices of appropriate sizes. Then $A^{(d)} > 0$.

Proof. Write

$$A = A^2A^{(d)} + N, \quad \text{where} \quad N = A - A^2A^{(d)}.$$

It can be easily verified that

$$(A^2A^{(d)})^{\#} = A^{(d)}$$

and N is a nilpotent matrix of (nilpotency) index k. Thus $B = A^2A^{(d)} > 0$ and $B^{\#} > 0$. Also, $A^{(d)} = \sum_i \alpha_i A^{m_i}$, $m_i > k$, $\alpha_i \neq 0$ gives

$$B^{\#} = A^{(d)} = \sum_i \alpha_i A^{m_i} = \sum_i \alpha_i (A^2A^{(d)})^{m_i}$$

$$= \sum_i \alpha_i B^{m_i}.$$

Thus by Theorem A there exists a permutation matrix P such that

$$PBP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are nonnegative matrices and J is a direct sum of matrices of type (I) or (II) (not necessarily both).

Let $J = \Sigma J_i$, where

$$J_i = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & \ddots & & & \\ & & & S_i & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}.$$

Then $PBP^T = \Sigma B_i$, $B_iB_j = 0$, $i \neq j$, where

$$B_i = \begin{bmatrix} J_i & J_iD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ_i & CJ_iD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also

$$B_i^{\#} = \begin{bmatrix} J_i^{\#} & J_i^{\#}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ_i^{\#} & CJ_i^{\#}D & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From Theorem A it immediately follows that

$$S_i^{\#} = \begin{cases} \beta^{-2}S_i & \text{if } S_i \text{ is of type (I).} \\ (\beta_{12}\cdots\beta_{d1})^{-1}S_i^{d-1} & \text{if } S_i \text{ is of type (II).} \end{cases}$$

But then $J_i^{\#} = \beta^{-2}J_i$ or $J_i^{\#} = (\beta_{12}\cdots\beta_{d1})^{-1}J_i^{d-1}$. This gives

$$B_i^{\#} = \beta^{-2}B_i \quad \text{or} \quad B_i^{\#} = (\beta_{12}\cdots\beta_{d1})^{-1}B_i^{d-1}.$$

Since $B_i \geqslant 0$, $B_i^{\#} \geqslant 0$, we apply Theorem A to the matrix B_i and obtain that for some permutation matrix P_i

$$P_i B_i P_i^T = \begin{bmatrix} G & GD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CG & CGD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are nonnegative matrices of appropriate sizes and G is a direct sum of matrices of type (I) or (II) (We might mention that the matrices C, D occurring above in the form for $P_i B_i P_i^T$ are not the same as the ones occurring earlier in the form for B_i). As stated in Theorem A, the summands of G are determined by polynomial $p(t)$ where $B_i^{\#} = p(B_i)$.

Now, we have

$$B_i^{\#} = \beta^{-2} B_i \quad \text{or} \quad B_i^{\#} = (\beta_{12} \cdots \beta_{d1})^{-1} B_i^{d-1}$$

Thus

$$p(t) = \beta^{-2} t \quad \text{or} \quad p(t) = (\beta_{12} \cdots \beta_{d1})^{-1} t^{d-1}.$$

Therefore, summands of type (I) are $\gamma x y^T$, where γ is a positive root of $\beta^{-2} t^2 = 1$. This gives $\gamma = \beta$. And summands of type (II) are

$$\begin{bmatrix} 0 & \gamma_{12} x_1 y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \gamma_{23} x_2 y_3^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where γ_{ij} are arbitrary positive numbers such that their product $\gamma_{12} \cdots \gamma_{d1}$ is a root of

$$(\beta_{12} \cdots \beta_{d1})^{-1} t^{(d-1+1)/d} = 1.$$

This gives

$$\gamma_{12} \cdots \gamma_{d1} = \beta_{12} \cdots \beta_{d1}.$$

Further, since

$$\operatorname{rank} G = \operatorname{rank} B_i = \operatorname{rank} J_i = \operatorname{rank} S_i,$$

we get that G consists of only one summand of type (I) or (II) and this summand is of the same form as the S_i.

From $A=B+N$ and $PBP^T=B_1+\cdots+B_r$ we have

$$A=P^TB_1P+\cdots+P^TB_rP+N.$$

If we set

$$A_i=P^TB_iP,$$

then

$$A=A_1+\cdots+A_r+N,$$

where $A_i \geqslant 0$; $A_iA_j=0$, $i\neq j$;

$$A_iN=0=NA_i;$$

and for some permutation matrix P_i,

$$P_i^TA_iP_i=\begin{bmatrix} G & GD & 0 & 0\\ 0 & 0 & 0 & 0\\ CG & CGD & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

where G is a matrix of type (I) or (II). This completes the proof in one direction.

Conversely, we have

$$A=A_1+\cdots+A_r+N,$$

where $A_i \geqslant 0$; $A_iA_j=0$, $i\neq j$; etc. We may delete any A_i which is zero. So we can assume that either $A=N$ or $A=A_1+\cdots+A_r+N$, where all $A_i\neq 0$. In the former case $A^{(d)}=0$, and hence the converse is trivial. In the latter case, no A_i can be invertible. Thus $A_i^{\#}$ is a polynomial in A_i with zero constant term. Therefore,

$$A_i^{\#}A_j=0, \qquad i\neq j.$$

This immediately yields

$$(A_1+\cdots+A_r)^{\#}=A_1^{\#}+\cdots+A_r^{\#}.$$

Also, it is clear by hypothesis that $A_i^{\#}\geqslant 0$. Therefore $(A_1+\cdots+A_r)^{\#}\geqslant 0$. Using $N^k=0$, one verifies directly that

$$A^{(d)}=(A_1+\cdots+A_r)^{\#}. \tag{1}$$

Hence $A^{(d)}\geqslant 0$ as desired. ■

Incidentally, we may mention that it follows from (1) that $A_1+A_2+\cdots+A_r=A^2A^{(d)}$.

The sum $A^2A^{(d)}=A_1+A_2+\cdots+A_r$ in the decomposition of A given in Theorem 3 will be called the nilpotent-free part of A, and N the nilpotent part of A.

The next theorem shows that if A is a row (or column) stochastic matrix with $A^{(d)}\geqslant 0$, then $A^{(d)}$ is some power of A, and the nilpotent-free part of A is also row (or column) stochastic.

THEOREM 4. *Let A be a row (or column) stochastic matrix with $A^{(d)}\geqslant 0$. Then $A^{(d)}=A^u$ for some positive integer u. The nilpotent-free part $B=A^2A^{(d)}$ is also row (or column) stochastic, and the nilpotent part N is such that the sum of entries in each row (or column) is zero.*

Proof. Let A be of index k. Then

$$A^{(d)}=(A^{(d)})^kA^{k-1}=(A^k)^{(d)}A^{k-1}$$

$$=(A^k)^{\#}A^{k-1}=A^{k\prime}A^{k-1}, \qquad \text{by [8, Corollary 4]},$$

since A^k is also row (or column) stochastic. Hence $A^{(d)}=A^u$ for some positive integer $u>k$. Then $B=A^2A^{(d)}=A^{u+2}$ is also row (or column) stochastic, and hence the sum of entries in each row (or column) of N is zero. ■

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