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Quotient rings of algebras of functions and operators

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In felicitation to our teachers P.B. Bhattacharya and F. Smithies

Abstract. Various quotient rings of rings B of Banach algebra A-valued continuous functions on a completely regular Hausdorff Space X are constructed in terms of continuous functions defined on dense open subsets of X taking values in the maximal quotient ring of the Banach algebra A. This extends the results proved by N. J. Fine, L., Gillman and J. Lambek (1965) for the case of A, the field of real numbers. The pattern is similar and utilizes as well as generalises the results proved for algebras of multipliers of B by C. A. Akemann, G. K. Pedersen and J. Tomiyama (1973). The techniques combine those from algebra, analysis and topology. The details of the cases when A is the normed division algebra of real quaternions or the operator algebra $B(H)$ of a Hilbert space H are given to illustrate our results.

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1 Introduction

Quotient rings, particularly, the maximal (i.e., complete) right quotient ring $Q_r(R)$ of an abstract ring R with no total left zerodivisors were initially defined and studied by R. E. Johnson [13], Y. Utumi [22] and G. D. Findlay and J. Lambek [6]. R. E. Johnson [14], J. Lambek ([16], [17]) give a lucid account of quotient rings to which we shall often refer. Roughly speaking,

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quotients are equivalence classes of module homomorphisms defined on algebraically dense ideals D of R into R, the name arising from the fact that the quotient $3/7$, for instance, can be thought of as the module homomorphism ϕ defined on the ideal $7\mathbf{Z}$ of $\mathbf{Z}$ into $\mathbf{Z}$ by $\phi(7n) = 3n$, or equivalently, ψ defined on the ideal $14\mathbf{Z}$ of $\mathbf{Z}$ into $\mathbf{Z}$ by $\psi(14n) = 6n$, and so on ... Various quotient rings of the ring $C(X)$ of real-valued continuous functions on a completely regular Hausdorff space X were determined by N. J. Fine, L. Gillman and J. Lambek ([7], Chapter 2) and well-studied in later chapters. The methods indeed apply to complex-valued functions with obvious modifications or even real quaternion-valued functions with some extra effort. Further study of quotient rings of such rings of continuous functions was not much pursued by algebraists. Some Functional analysts, Harmonic analysts and Topologists studied special quotients known as multipliers or centralizers or even module homomorphisms defined on a Banach algebra, particulary a C^*-algebra or a group algebra R or a topologically dense ideal D of R and also related problems like adjoining inverses to Banach algebras or certain rings or lattices of continuous functions. The literature is too vast to quote here. See, for instance, B. E. Johnson [12], R. C. Busby [4], A.W. Hager [8], D. C. Taylor [21], E. Hewitt and K. A. Ross [10], R. Larsen [19], J. Wood [24] and J. Esterle [5] to get a general idea. We simply refer to the paper by C. A. Akemann, G. Pedersen and J. Tomiyama[1], who studied multipliers of C^*-algebras of continuous cross sections of a fibred space, a special case of which is the C^*-algebra of continuous functions on a locally compact Hausdorff space X that take values in a C^*-algebra and vanish at infinity.

Theorem 1.1. ([1], Corollary 3.4) *Let A be a C^*-algebra and $B = C_0(X, A)$, the C^*-algebra of continuous A-valued functions on a locally compact Haussdorff space X that vanish at infinity. Let $M(A)$ be the multiplier algebra of A (i.e., the largest C^*-subalgebra of the second dual A'' of A in which A is an ideal). Let β be the strict (i.e., strong) topology on $M(A)$ generated by seminorms of the form $||b|| = ||ba|| + ||ab||$, where b is in $M(A)$ and a is in A. Then $M(B)$ can be identified with the algebra of bounded continuous functions on X to $(M(A), \beta)$.*

We next quote a result of Utumi (cf. [17], Proposition 4.3.9).

Theorem 1.2 (Utumi). *The maximal right quotient ring of the product of rings with no total left zerodivisors is the product of their maximal right quotient rings.*

Taking all the rings to be a Banach algebra A with no total left zerodivisors and X, a discrete topological space, and equipping $Q_r(A)$ with any topology, we may reformulate it as:

$$Q_r(C(X, A)) = C(X, Q_r(A)).$$

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This paper aims at determining $Q_r(R)$ or its special subsets for different rings R of continuous functions on a completely regular Hausdorff space X to certain Banach algebras A and thus generalise Theorem 1.1 and topologise Theorem 1.2. The basic pattern is the same as that of N. J. Fine, L Gillman and J. Lambek [7] and we utilize Theorem 1.1 amongst other things! For instance, Banach algebra $\mathbf{H}$ of real quaternions and the operator algebra $B(H)$ of a Hilbert space H satisfy the requirements needed on A. So, their quotient rings are presented as well.

2 Preliminaries and motivation

We collect a few basic known definitions, notation, results and remarks for further use and for all others we refer the reader to standard sources such as those given in the list of references.

2.1.
Let R be a ring with centre $Z(R)$ and no total left zerodivisors, i.e., for an r in R, $rR = 0$ implies $r = 0$. Let $G(R)$ be the set of elements of A which are not zerodivisors. Let $\mathbf{Z}$ be the ring of integers, $\mathbf{H}$, the division ring of real quaternions, $\mathbf{R}$ and $\mathbf{C}$, the fields of real and complex numbers respectively. We reformulate the relevant part of ([22], Sect. 1) and ([17], Chapters 2 and 4) as follows.

(a) Let R be a subring of a ring S. We say that S is a *right quotient ring* of R if for any pair (s,t) of elements in S, s not zero, there exists an r in R such that sr is not zero and tr is in R. In particular, R is a right quotient ring of itself.
(b) Let T be a subring of R and R, a subring of a ring S. Then R is a right quotient ring of T and S is a right quotient ring of R if and only if S is a right quotient ring of T.
(c) Suppose R has no left identity and let R_1 be its unitization. If for an r in R and an integer m, there is an r' in R with $mr' = r$, then R_1 is a right quotient ring of R. In particular, it is so, if R is a real or complex algebra.
(d) In view of (b) and (c) above, if R is an algebra with no left identity, then $Q_r(R)$ and $Q_r(R_1)$ coincide.
(e) A right ideal D in R will be called *dense* if R is a right quotient ring of D. Let $\mathcal{D}_r(R)$ denote the set of all dense right ideals of R. $\mathcal{D}(R)$ will denote the subcollection consisting of (two-sided) ideals in $\mathcal{D}_r(R)$. For a dense right ideal D of R, let $Hom_R(D,R)$ denote the additive group of all right R-homomorphisms ϕ (also known as left centralizers or multipliers in the literature) of D to R. Let $\mathcal{H}_R$ denote the union of all such $Hom_R(D,R)'s$. Let ϕ and ϕ' be in $\mathcal{H}_R$ with domains D and D'

respectively. If $D \cap D'$ contains a dense right ideal D'' of R such that ϕ and ϕ' agree on it, we say that ϕ is *equivalent* to ϕ' and write $\phi\,\theta\,\phi'$. On the other hand, if D' is a right quotient ring of $\phi(D)$, then we may define the right R-homomorphism $(\phi'\,\phi)$ on D. Finally, if $D = D'$, then we can define $\phi + \phi'$ on D. The set $Q_r(R)$ of equivalence classes of $\mathcal{H}_R$ by the relation θ can then be made into a ring and is called the *maximal (or, complete) right quotient ring* of R.

(f) A right ideal D of R is dense if and only if for r, r' in R, with r not zero, there is an r'' in R satisfying rr'' not zero and $r'r''$ in D. Also, if D is a (two-sided) ideal then the condition reduces to: r in R, $rD = 0$ implies $r = 0$. Further, intersection of two dense right ideals in R is dense in R. This simplifies the construction of $Q_r(R)$ given in (e).

(g) If R contains a smallest dense right ideal D, then $Q_r(R)$ is isomorphic to $Hom_R(D, D)$.

(h) Suppose R is commutative then the right and left counterparts of each of the notions defined above coincide and, therefore, we can drop the usage of right or left. We now reformulate the relevant part of [5, section 2]. Suppose $G(R)$ is not empty. For $s \in G(R)$ and $r \in R$, let $D_{r,s} = \{s' \in R : rs' \in sR\}$. For $s' \in D_{r,s}$, there is a uniqe $r' \in R$ with $rs' = sr'$ and thus we may define a map $\varphi_{r,s}$ on $D_{r,s}$ to R by $\varphi(s') = r'$. $\varphi_{r,s}$ is called a *semimultiplier* of R. The set $SM(R)$ of semimultipliers of R can be identified with the ring of fractions of R (with denominators in $G(R)$ and numerators in R), also known as the *classical quotient ring* of R and denoted by $Q_{cl}(R)$ defined in 2.3(b), in general.

2.2

(a) R is called *prime* if r, s in R and $rRs = 0$ imply $r = 0$ or $s = 0$.

(b) (Utumi)([22], Sect. 5). Suppose R is prime with nonzero socle D and Re is its minimal left ideal. Then $Q_r(R)$ is isomorphic to $Hom_R(D, D)$, which, in turn, is isomorphic to the ring of linear transformations of the right vector space Re over eRe.

2.3 Suppose R has an identity. (cf. [17], Chapters 2 and 4).

(a) Let R be a subring of a ring S. S is called a *classical ring of right quotients* of R if and only if all non-zero-divisors of R have inverses in S and all elements of S have the form $r'r^{-1}$, where, r, r' are in R and $r \in G(R)$ (i.e, r is not a zerodivisor of R).

(b) R has a classical ring of right quotients S if and only if it satisfies the following condition, known as right *Ore's condition.* For r, s in R with $s \in G(R)$, there exist r', s' in R with $s' \in G(R)$, such that $rs' = sr'$. In this case, S is isomorphic to $Q_{cl,r}(R)$, the subring of $Q_r(R)$ consisting of elements of the form rs^{-1}, r and s in R with $s \in G(R)$, (the principal right ideal sR and the right R-homomorphism sending sr' to rr', being a suitable representative). $Q_{cl,r}(R)$ is called the *classical ring of right quotients of R*.

(c) *The symmetric maximal quotient ring*, $Q_\sigma(R)$, is defined as in (cf. [23], p. 172 and [18]) i.e., $Q_\sigma(R) = \{q \in Q_r(R) : Dq \subset R$ for some dense left ideal D of $R\}$ (see also [11] for further study).

2.4 Let A be a real or complex Banach algebra, which has no total left zerodivisors and satisfies the condition that for a in A, $||a|| = \sup\{||ab|| : b$ in $A, ||b|| \leq 1\}$.

(a) A Banach algebra with an identity of norm one or a right approximate identity bounded by one satisfies the requirements on A. Thus, we may take A to be the operator algebra $B(H)$ of a Hilbert space H, or, a C^*-algebra, or, the Banach algebra $C_b(X)$ of bounded continuous real or complex functions on a completely regular Hausdorff space X. Trivial cases are $A = \mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$.

(b) Members of $Hom_A(A, A)$ are called left multipliers, and the left multiplier algebra, $M_l(A)$ is simply $Hom_A(A, A)$. It is a right quotient ring of A and has the identity operator I on A as the identity.

(c) ([1], [4], [21]). For a C^*-algebra A, the multiplier algebra $M(A)$ of A is defined to be (cf. [1], Sect. 1) the largest C^*-subalgebra of the second dual A'' (with Arens product) of A in which A is an ideal. The abstract algebra $M(A)$ can also be equipped with the strict topology β generated by the seminorms of the form $||b||_a = ||ab|| + ||ba||, a \in A$. $M(A)$ is a right quotient ring of A and has the identity operator I on A as the identity. In view of Remarks in 2.1 and 2.2 above, we have $Q_r(A) = Q_r(M_l(A)) = Q_r(M(A))$.

(d) The second author is studying the relationship between A'' and $Q_r(A)$ for different A's, particularly, group algebras and hypergroup algebras.

(e) Suppose A is commutative. By [5], Proposition 2.14 every semimultiplier of A is closed.

This inspired our next theorem. We thank G. R. Allan for bringing [5] to our notice.

Theorem 2.5. *Let A be a Banach algebra satisfying the conditions set in 2.4. Suppose A is commutative.*

(a) Let D be a dense ideal in A and $\varphi \in Hom_A(D, A)$.

(i) φ is closable.

(ii) The closure ψ of φ is in $\mathcal{H}_A$ and $\psi\, \theta\, \phi$.

(iii) If D is (topologically) closed then ϕ is bounded.

(iv) If ϕ is closed and bounded then D is closed.

(b) $Q(A) = \{[\phi] : \phi \in \mathcal{H}_A$ *and* ϕ *is closed*$\}$.

Proof.

(a) Let, if possible, ϕ be not closable. Then there is a sequence (a_n) in D with $a_n \to 0$ and $\phi(a_n) \to b \neq 0$. Because D is a dense ideal in A, there exists a $c \in D$ with $bc \neq 0$. But $bc = \lim_{n\to\infty} \phi(a_n c) = \lim_{n\to\infty} \phi(c)a_n = 0$, a contradiction. Hence ϕ is closable . This gives (i).
For (ii) we note that the domain D_1 of ψ is a linear space and if $a \in D_1$ and $b \in A$ then there is a sequence (a_n) in D with $a_n \to a$ and $\varphi(a_n) \to \psi(a)$. So $a_n b \to ab$ and $\varphi(a_n b) = \varphi(a_n)b \to \psi(a)b$. Thus, $ab \in D_1$ and $\psi(ab) = \psi(a)b$. So D_1 is a dense ideal in A and $\psi \in Hom_A(D_1, A)$. Also $\phi\,\theta\,\psi$. This shows (ii).
For (iii) we first note that with D, ψ, D_1 as in the proof of (ii) above, we have $D_1 = D$ and ψ is closed. So, by the Closed Graph Theorem, ψ is bounded. Thus, φ is bounded.
(iv) It is a standard fact for linear operators defined on subspaces of a Banach space to itself.

(b) It follows immediately from (a) (ii). □

2.6. Let X be a completely regular Hausdorff space and A as in 2.4 above. Let $B = C(X, A)$, the topological algebra of continuous functions on X to A with pointwise operations and the topology of uniform convergence on a collection $\mathcal{A}$ of subsets of X, whose union is X.

For a scalar-valued continuous function h on X and an a in A, the function ha on X to A defined by $(ha)(x) = h(x)a$ is in $C(X, A)$. If A has an identity 1_A then $h1_A$ will be abbreviated to h only.

(a) B has an identity if and only if A does.
(b) $Z(B) = C(X, Z(A))$.
(c) Let B_b be the subalgebra $C_b(X, A)$ of B consisting of bounded functions in B with $\mathcal{A}$ consisting of X alone. Thus B_b can be made into a Banach algebra with the supnorm $||.||$, i.e., for g in B_b, $||g|| = \sup\{||g(x)|| : x \in X\}$. As in (a) above, B_b has an identity if and only if A does.
(d) Because A has no total left zerodivisors, for any g in B and x in X with $g(x)$ not equal to zero, there is an a in A with $g(x)a$ not equal to zero. Because of complete regularity of X, for any neighbourhood W of x we have a continuous function h on X to $[0, 1]$ with $h(x) = 1$ and $h = 0$ outside W. Thus $g_1 = ha$ is in B_b. Also gg_1 is not zero. As a consequence, B and B_b have no total left zerodivisors. Because each g_2 in B is locally bounded, this also helps to show that B is a right

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quotient ring of B_b. We note a significant consequence of this, which can be compared with ([7], Theorem 2.3).

Theorem 2.7. *The maximal right quotient rings of $C(X, A)$ and $C_b(X, A)$ coincide.*

2.8. Let X be a locally compact Hausdorff space and A be as in 2.4 above. Let B_o be the Banach subalgebra $C_o(X, A)$ of B_b consisting of functions in B_b that vanish at infinity and let B_{oo} be the (topologically) dense subalgebra $C_{00}(X, A)$ of B_o consisting of functions in B that have compact support.

(a) B_o is an ideal of B_b and B_{oo} is an ideal of B.
(b) In 2.6 (d) above we can take the neighborhood W of x to be compact and thus have g_1 in B_{oo}. As a consequence, both B_o and B_{oo} have no total left zerodivisors and have a left identity if and only if A does and X is compact.
(c) In case A has a right identity of norm one, B_o has a right approximate identity in B_{oo} bounded by one, and, thus, B_o satisfies the requirements on A as in 2.4.
(d) (b) also gives that B_{oo} is a dense right ideal in B.

We now note an important application of the results in [1], particularly, Theorem 1.1.

Theorem 2.9. *Suppose X is locally compact.*

(a) The maximal right quotient rings of B, B_b , B_o and B_{oo} all coincide.
(b) Let A be a C^*-algebra.
Then $Hom_{B_o}(B_o, B_o) = M(B_o) = C_b(X, (M(A), \beta))$ is a quotient ring of B_o. Further, the maximal right quotient rings B_o, $C_b(X, M(A))$ and that of $C_b(X, (M(A), \beta))$ all coincide.

2.10. Thus, to study the problem of determining the maximal right quotient rings of algebras of continuous functions on a completely regular Hausdorff space X taking values in A as in 2.4 above, it is enough to confine our attention to the algebra of bounded operators on X to a Banach algebra A with identity. Of course, individual study would be required for different levels of such quotient rings, which we do not take up in this paper (cf. [20], for the real-valued functions).

2.11. The basic pattern of [7] will be adhered to as long as possible, which we now indicate. Let X be a completely regular Hausdorff space. Let A be a Banach algebra with identity of norm one. Let $\mathcal{U}$ be the class of dense open subsets of X. Then $\mathcal{U}$ is closed with respect to nonempty arbitrary unions and finite intersections. For $U \in \mathcal{U}$, let R_U denote the ring $C(U, A)$. Let $\mathcal{F}$ denote the class of pairs (f, U) with $U \in \mathcal{U}$ and $f \in R_U$. For (f, U) and

(g, V) in $\mathcal{F}$, $(f, U) + (g, V)$ and $(f, U)(g, V)$ will denote the functions on $U \cap V$ obtained by pointwise sum and product respectively.

(a) For $U, V \in \mathcal{U}$ with $V \subset U$, the restriction map $\rho_{V,U}$, defined on R_U to R_V by $\rho_{V,U} f = f/V$, is injective.

(b) If (f, U) and (g, V) are in $\mathcal{F}$ and satisfy $f = g$ on $U \cap V$, then h defined on $U \cup V$ by $h = f$ on U and $h = g$ on V is in $R_{U \cup V}$. Thus, we may define an equivalence relation $\sim$ on $\mathcal{F}$ by $(f, U) \sim (g, V)$ if and only if $f = g$ on $U \cap V$ if and only if there is an $(h, W) \in \mathcal{F}$ with $U \subset W, V \subset W$ such that $f = h$ on U and $g = h$ on V.

(c) For $(f, U) \in \mathcal{F}$, let $[(f, U)]$ denote the equivalence class of (f, U). Let $U_o = \cup\{V : (g, V) \in [(f, U)]\}$ and f_0 on U_0 be given by $f_0 = g$ on V for $(g, V) \in [(f, U)]$. Then $(f_0, U_0) \in [(f, U)]$ is the unique member of $[(f, U)]$ with the largest domain. Thus we may identify $[(f, U)]$ with (f_0, U_0) and denote it by $(f, U)_0$, or, simply by f_0.

(d) For an $(f, U) \in \mathcal{F}$ and $g = (g, X) \in R_X = B$, we shall use the abbreviation $f\, g \in B$ for the condition that $((f, U)(g, X))_0$ has domain X and write $((f, U)(g, X))_0$ as $f\, g$ only.

(e) Let $S = S(X, A) = \{(f_0, U_0) : (f, U) \in \mathcal{F}\}$. S can be made into a ring if we define the sum and product as below:

$$(f, U)_0 + (g, V)_0 = ((f, U) + (g, V))_0,$$

$$(f, U)_0 (g, V)_0 \;\; = ((f, U)(g, V))_0.$$

Further, no confusion can arise if, for an $(f, U) \in S$, we use the symbol f alone. With this understanding, $B = R_X$ is a subring of S.

(f) Let $S_b = S_b(X, A)$ denote the subring of $S = S(X, A)$ consisting of bounded functions.

(g) For any right ideal D of $B = C(X, A)$ or of $B_b = C_b(X, A)$, the *cozero set of* D is the set $U = \{x \in X : g(x) \neq 0$ for some g $\in D\}$. Clearly, U is open.

(h) Results in [7], Chapter 2 relevant to our problem can be summarized as: $S(X) = S(X, \mathbf{R})$ is the maximal quotient ring of $C(X)$ as well as of $C_b(X)$.

(i) The gist of [9] is that $Q(C(X))$ is isomorphic to some $C(Z)$ with Z a completely regular Hausdorff space if and only if the set of isolated points of X is dense in X (provided no measurable cardinals exist nearby).

3 Quotient rings of algebras of Banach algebra-valued continuous functions

In this section, R will be a ring with identity, A, a Banach algebra with identity 1_A of norm 1, and X, a completely regular Hausdorff space. The notation and terminology will be as in Sect. 2, unless stated otherwise. For instance, B_b will be the Banach algebra $C_b(X, A)$ with the constant function 1_A as the identity. We begin with a few basic concepts and results.

Discussion and definitions 3.1

(i) For a $D \in \mathcal{D}_r(A)$, $Hom_{b,A}(D, A)$ denotes the set of bounded right A-homomorphisms on D to A. We note that, for a $q \in Q_r(A)$, $q \cap Hom_{b,A}(D, A)$ is either empty or a singleton. If $\varphi \in Hom_{b,A}(D, A)$ then its unique continuous extension $\bar{\varphi}$ to $\overline{D}$ in $Hom_{b,A}(\overline{D}, A)$ and $[\varphi] = [\bar{\varphi}]$. $\bar{\mathcal{D}}_r(A)$ will denote the collection of (topologically) closed dense right ideals of A. Similar notation will be used for other entities.

(ii) The *maximal topological right quotient ring* of A is the union of the sets of equivalence classes of elements in $Hom_{b,A}(D, A)$ as D varies over $\bar{\mathcal{D}}_r(A)$ and it will be denoted by $Q_{b,r}(A)$. The subset $Q_{b,tr}(A)$ of $Q_{b,r}(A)$, when D varies only on ideals in $\bar{\mathcal{D}}(A)$ will be called the *maximal topological bi-right quotient ring of* A.

(iii) It follows form Theorem 2.5 that if A is commutative then $Q_b(A) = \{[\varphi] : \varphi \in Hom_A(D, A),\ D \in \bar{\mathcal{D}}(A)\}$.

(iv) For a D in $\mathcal{D}_r(A)$, the *strict topology*, β, on $Hom_A(D, A)$ is the topology of pointwise convergence on D. The operator norm $||.||$ on the space of bounded operators on D to A induces the *uniform operator topology* on $Hom_{b,A}(D, A)$. For a $q = [\phi] \in Q_{b,r}(A)$, we shall write $||q||$ for the infimum of $||\psi||$, such that ψ is in $Hom_{b,A}(D', A), D' \in \mathcal{D}_r(A)$ and $\phi\ \theta\ \psi$, i.e. φ is equivalent to ψ.

(v) Let E be a right ideal of B_b. For an $x \in X$, E_x denotes the set $\{g(x) : g \in E\}$. It is clearly a right ideal. For $\emptyset \neq Y \subset X$, let $E_Y = \cup_{x \in Y} E_x$. The *co-zerodivisor set of* E, denoted by $CozdE$, is the set $V = \{x \in X :$ there is a $g \in E$ with $g(x)$ not a right zerodivisor$\}$. Clearly, if E is an ideal then $CozdE \subset \{x \in X : E_x$ is a dense right ideal of $A\}$.

Theorem 3.2. *Let* E *be a right ideal of* B_b, U *its cozero set,* $Coz(E)$ *and* V *its cozerodivisior set,* $Cozd(E)$.

(a) If E *is a dense right ideal in* B_b *then* U *is dense in* X.
(b) For $\emptyset \neq Y \subset X, E_Y$ *is a right ideal in* A, *which is non-zero if and only if* $Y \cap U \neq \emptyset$.
(c) Let W *be a non-empty open subset of* X. *If* E *is a dense right ideal in* B_b, *then* E_W *is a dense right ideal of* A.
(d) If E *is an ideal and* V *is dense in* X *then* E *is a dense right ideal of* B_b.
(e) For $g \in E, x \in X$ *and* $\epsilon > 0$ *there is an* $h \in E$ *with* $h(x) = g(x)$ *and* $||h|| < ||g(x)|| + \epsilon$.
(f) For g', g'' *in* $E, x \in X$ *with* $g'(x) = g''(x)$ *and* $\epsilon > 0$, *there is an* $h' \in C(X)$ *with* $0 \leq h' \leq 1$, $h'(x) = 1$ *and* $||(g' - g'')h'|| \leq \epsilon$.

Proof.

(a) Let, if possible, U be not dense in X. Then there is a non-empty open subset W of X which is disjoint from U. Because X is completely regular, there exists a non-zero continuous function h on X to the closed interval $[0,1]$ with h zero outside W. Thus $hE = 0$. This contradicts the assumption that E is dense in B_b.

(b) It is trivial.

(c) Let $a, b \in A$ with a not zero. By (a), U is dense in X. So there is a point x common to U and W. Next, complete regularity of X gives a continuous function h on X to $[0,1]$ with $h(x)$ non zero but h zero outside $W \cap U$. Let $g_1 = ha$ and $g_2 = hb$. Then g_1 is not zero. Because E is a dense right ideal, there is a $g \in B_b$ with $g_1 g$ not zero and $g_2 g \in E$. This gives us a $y \in W$ with $g_1(y)g(y)$ not zero. So, $h(y)ag(y)$ is not zero and $h(y)bg(y)$ is in E_W. We set $c = g(y)$ and note that ac is not zero and bc is in E_W. This finishes the proof.

(d) Let $g \in B_b$ with g not zero. Then there is an x in V with $g(x) \neq 0$. So, for $a \neq 0$ in A, $ag(x) \neq 0$. Thus, for $a \neq 0$, $aE_x \neq 0$. This gives $gE \neq 0$. Because E is an ideal, the result follows from 2.1 (f).

(e) Because g is continuous, there is an open neighbourhood W of x with $||g(y) - g(x)|| < \epsilon$ for $y \in W$. By complete regularity of X, there is $h' \in C(X)$ with $h'(x) = 1$, $0 \leq h' \leq 1$ and h' zero outside W. Let $h = gh'$. h is then a desired member of E.

(f) Put $g = g' - g''$ in the proof of (e) above. □

Theorem 3.3. *Let E be a dense right ideal of B_b and U be its cozero set. Let $\phi \in Hom_{b,B_b}(E, B_b)$.*

(a) There is a unique bounded $f \in \prod_{x\in U} Hom_{b,A}(E_x, A)$ such that for x in U, g in E, $f(x)(g(x)) = \phi(g)(x)$. Also $||f|| = \sup_{x\in U} ||f(x)|| \leq ||\phi||$.

(b) For an open subset W of U, let $D_W = \cap_{x\in W} E_x$ and let f_W be the function on W to $Hom_{b,A}(D_W, A)$ given by $f_W(x) = f(x)/D_W$. Then f_W is β-continuous.

(c) $\phi \theta 0$ if and only if there is a dense open subset W of U and a dense right ideal $E' \subset E$ with $f(x) = 0$ on E'_x for $x \in W$.

Proof.

(a) Let $x \in U$, $g', g'' \in E$ be such that $g'(x) = g''(x)$. We claim that $\phi(g')(x) = \phi(g'')(x)$. Suppose not. Then $\epsilon = \frac{1}{||\phi||+1}||\phi(g')(x) - \phi(g'')(x)|| > 0$. So, by Theorem 3.2 (f), we have an $h' \in C(X)$ with $h'(x) = 1, 0 \leq h' \leq 1$ and $||(g' - g'')h'|| \leq \epsilon$. Thus, $||\phi(g')(x) - \phi(g'')(x)|| = ||(\phi(g')h' - \phi(g'')h')(x)|| = ||\phi((g' - g'')h')(x)|| \leq ||\phi((g' - g'')h')|| \leq ||\phi||.||(g' - g'')h'|| < ||\phi(g')(x) - \phi(g'')(x)||$, a contradiction. This settles our claim. Therefore, we can define, for $x \in U$, $g \in E$, $f(x)(g(x)) = \phi(g)(x)$. Because ϕ is a right B_b-homomorphism on E to A, we can easily show that $f(x)$ is a right A-homomorphism on E_x to A. Let $x \in U$, $g \in E$ and $a = g(x)$. Let $\epsilon > 0$ be arbitrary. Then, by Theorem 3.2 (e) there is an $h \in E$ with $h(x) = g(x) = a$, and $||h|| \leq ||a|| + \epsilon$. Thus, $||f(x)(a)|| = ||\phi(h)(x)|| \leq ||\phi||(||a|| + \epsilon)$. So, $||f(x)(a)|| \leq ||\phi||.||a||$. This gives $||f(x)|| \leq ||\phi||$. Therefore, $||f|| = \sup_{x \in U} ||f(x)|| \leq ||\phi||$.

(b) Let $x \in W$, $a \in D_W$ and $\epsilon > 0$ be arbitrary. For $y \in W$, there is a $g_y \in E$ with $g_y(y) = a$. Because g_x and $\phi(x)$ are continuous at x, there is an open neighbourhood W_1 of x such that $W_1 \subset W$ and $||g_x(z) - g_x(x)|| < \epsilon$, $||\phi(g_x)(z) - \phi(g_x)(x)|| < \epsilon$ for $z \in W_1$. Let $y \in W_1$. Then g_y is continuous at y. So, there is an open neighbourhood W_y of y with $W_y \subset W_1$ and $||g_y(z) - g_y(y)|| < \epsilon$ for z in W_y. As $g_y(y) = g_x(x) = a$, we have $||g_y(z) - g_x(z)|| < 2\epsilon$ for z in W_y. By complete regularity of X, we have an $h_y \in C(X)$ with $h_y(y) = 1$, $0 \leq h_y \leq 1$ and h_y zero outside W_y. Therefore, $||(g_y - g_x)h_y|| \leq 2\epsilon$. So $||\phi((g_y - g_x)h_y)|| \leq 2||\phi||\epsilon$. In particular, $||\phi(g_y - g_x)(y)|| \leq 2||\phi||\epsilon$, i.e., $||\phi(g_y)(y) - \phi(g_x)(y)|| \leq 2\epsilon||\phi||$. As a consequence, $||f(y)a - f(x)a|| = ||\phi(g_y)(y) - \phi(g_x)(x)|| \leq ||\phi(g_y)(y) - \phi(g_x)(y)|| + ||\phi(g_x)(y) - \phi(g_x)(x)|| \leq \epsilon + 2\epsilon||\phi||$. As $y \in W_1$ is arbitrary, we conclude that $||f(y)a - f(x)a|| < \epsilon(1+2||\phi||)$ for $y \in W_1$. This shows that f_W is β-continuous.

(c) This is trivial. □

Remarks 3.4. Let D, D' be dense right ideals of A and U and V be dense open subsets of X. Let f be a function on U to $Hom_{b,A}(D, A)$. Let g be in $C(V, D)$ and h a function on V to $Hom_{b,A}(D', A)$. Let k, also denoted by fg, be the function on $U \cap V$ to A defined by $k(x) = f(x)(g(x))$. Let $f + h$ be the function on $U \cap V$ to $Hom_{b,A}(D \cap D', A)$ defined by $(f + h)(x) = f(x)/D \cap D' + h(x)/D \cap D'$.

In the special case $U = V = X$, the following well-known results are the key-steps in the proof of Theorem 1.1 and the proof for the general case is similar.

(i) (a) If f is locally $||.||$-bounded and β-continuous, then k is continuous.
(b) If f is norm-continuous then k is continuous.
(c) If k has a continuous extension k_1 on X to A, then it is unique. We shall say, as in 2.11 (d), that fg is in $C(X, A)$ and write k_1 as fg only.
(d) If f and g are bounded then so is k. If, also, k has a continuous extension k_1 on X, then $||k_1|| \leq ||f||.||g||$. We shall, as in 2.11 (d), express this by just saying that $fg \in B_b$ and $||fg|| \leq ||f||.||g||$.

(ii) If f and h are β- continuous (respectively, $||.||$-continuous, locally bounded) then so is $f+h$.

(iii) Suppose for $x \in U \cap V$, $h(x)D' \subset D$.

(a) Suppose f is locally bounded. If both f and h are β-continuous then so is fh.

(b) If both f and h are $||.||$-continuous then so is fh.

(c) If f and h are both locally bounded (respectively, bounded) then so is fh.

(iv) The spaces $C_{lb}(U, (Hom_{b,A}(D,D), \beta))$, $C(U, Hom_{b,A}(D,D), ||\cdot||))$, $C_b(U, (Hom_{b,A}(D,D), \beta))$ and $C_b(U, (Hom_{b,A}(D,D), ||\cdot||))$ are all algebras.

(v) $C_b(U, (Hom_{b,A}(D,D), ||\cdot||)) \subset C_b(U, (Hom_{b,A}(D,D), \beta))$ are both closed subalgebras of the normed algebra $\mathcal{B}_b(U, (Hom_{b,A}(D,D), ||\cdot||))$ of bounded functions with sup norm.

Theorem 3.5. *Let D be a dense right ideal of A and U, a dense open subset of X.*

(a) $C_{lb}(U, (Hom_{b,A}(D,D), \beta))$ is a right quotient ring of $C_b(X, D)$.

(b) $C(U, (Hom_{b,A}(D,D), ||\cdot||))$ is a right quotient ring of $C_b(X, D)$.

(c) Suppose D is an ideal in A. $C_b(U, (Hom_{b,A}(D,A), \beta))$ can be identified with a subset of the maximal topological right quotient ring of $C_b(X, A)$.

(d) Suppose X is locally compact and D is (topologically) closed in A. Then $C(U, (Hom_{b,A}(D,D), \beta))$ is a right quotient ring of $C_{oo}(X, D)$, of $C_0(X, D)$ and also of $C_b(X, D)$.

Proof. (a) Let f and f', f not zero, be elements of $C_{lb}(U, (Hom_{b,A}(D,D), \beta))$. There is an x in U such that $f(x)$ is not zero. Because D is dense right ideal in A, there is an a in D with $f(x)a$ not zero. Because f and f' are locally bounded, there is an open neighbourhood V of x contained in U on which both f and f' are bounded. As X is completely regular, there is an open neighboourhood W of x with V containing the closure of W and an h in $C(X)$ with $h(x) = 1$, $0 \leq h \leq 1$ and h zero outside W. We put $g = ha$, $k = fg$ and $k' = f'g$ on U. Then g is in $C_b(X, D)$ and k is not zero. By Remark 3.4 (i)(a), k' is continuous on U and is zero outside the closure of W. So, we may extend it continuously on X to D by just putting it equal to zero outside U. Thus, $f'g$ is in $C_b(X, A)$. This finishes the proof.

(b) It follows as in (a) above if we use Remark 3.4(i)(b) instead of Remark 3.4(i)(a).

(c) Let f be in $C_b(U, (Hom_{b,A}(D,A), \beta))$. Let E be the set of all those g in $C_b(X, D)$ for which fg is in $B_b = C_b(X, A)$. Then E is a right ideal in B_b. We show that it is dense in B_b. Let f' and f'' be in B_b with f''

not zero. Then there is an x in U with $f''(x)$ not zero. Because A has no total left zerodivisors, we have a b in A with $f''(x)b$ not zero. Also D is dense in A. So there is an a in D with $f''(x)ba$ not zero and ba in D. We may proceed as in the proof of (a) above with $f''(x)b$ taking the place of $f(x)$ and can even take $V = U$ and obtain h in $C(X)$. We put $g' = hba$. Then $(f''g')(x)$ is not zero and thus, $f''g'$ is not zero. Moreover, $f'g'$ is in B_b. Because D is an ideal in A we have $f'g'$ takes values in D. Also, as in the proof of (a) above, $f(f'g')$ can be extended to a continuous function on X by simply putting it zero outside U. It is clearly bounded. Therefore, $f(f'g')$ is in $C_b(X, A)$. This finishes the proof of denseness of E in B_b. As a by-product, it, combined with the proof of (a), also gives that we may define a right B_b-homomorphism ϕ on E to B_b by $\phi(g) = fg$ and ϕ is bounded, its norm being the same as that of f.

(d) Let $f \in C(U, (Hom_{b,A}(D, D), \beta))$. If W be a compact subset of U then $f(W)$ is a compact subset of $(Hom_{b,A}(D, D), \beta)$. So $f(W)$ is a pointwise bounded subset of $Hom_{b,A}(D, D)$.
Because D is a Banach space, the uniform boundedness principle gives that $f(W)$ is $||.||$-bounded. Because X is locally compact so is U. Therefore, f is locally bounded. Thus $C_{lb}(U, (Hom_{b,A}(D, D), \beta)) = C(U, (Hom_{b,A}(D, D), \beta))$. The result now follows from (a) and 2.8. □

Corollary 3.6.

(a) *$Hom_{b,B_b}(B_b, B_b)$, i.e., the algebra of bounded left multipliers can be identified with $C_b(X, (Hom_{b,A}(A, A), \beta)$.*

(b) *(cf. Theorem 1.1) Suppose X is locally compact. Then $Hom_{b,B_o}(B_o, B_o)$ and $Hom_{b,B_{oo}}(B_{oo}, B_{oo})$ can both be identified with $C_b(X, (Hom_{b,A}(A, A), \beta))$.*

Proof.

(a) Take $D = A$ and $U = X$ in Theorem 3.5 and $E = B_b$ in Theorem 3.3.

(b) Take $D = A$ and $U = X$ in Theorem 3.5 and $E = B_o$ and B_{oo} respectively in Theorem 3.3. □

Corollary 3.7. *Let D_1 be a dense ideal of $C(X)$ with cozero set U and D_2 be a dense right ideal of A, which is also an ideal. Then the vector subspace $Q_{b,D_1,D_2,r}(B_b)$ of the maximal right quotient ring $Q_r(B_b)$ consisting of elements arising from the right ideal E_{D_1,D_2} of B_b generated by ha with h in D_1 and a in D_2 can be made to correspond to the space $C_b(U, (Hom_{b,A}(D_1D_2, A), \beta))$.*

Proof. In this case, for all x in U, $(E_{D_1,D_2})_x$ contains D_2. □

Corollary 3.8. *The ring $S(X, A)$ defined in 2.11 is a right as well as a left quotient ring of B_b as well as of B. It is contained in the symmetric quotient ring $Q_\sigma(B_b)$ of B_b.*

Proof. Let U be a dense open subset of X. For each x in U and an open set W containing x with its closure contained in U, there is an h in $C(X)$ with $h(x) = 1$, $0 \leq h \leq 1$, h zero outside W. Let D_1 be the set of all such functions. Then D_1 is dense in $C(X)$ and has cozero set U. We take $D_2 = A$ in Corollary 3.7 above and then vary U over $\mathcal{U}$ to obtain $S_b(X, A)$ as a subring of the maximal topological right (as well as left) quotient ring of B_b. The proof can be completed on noting that by 2.6 (d) $S(X, A)$ is a right (as well as left) quotient ring of $S_b(X, A)$. □

Some applications 3.9.

(i) We choose the Banach algebra A to be a division algebra. We note that A can be only **R**, **C** or **H** and A has no non-zero right ideals other than A itself. Theorem 3.5 gives that $S_b(X, A)$ is the maximal right (as well as left) topological quotient ring of B_b. and the result truly resembles the one in ([7],Chapter 2) for $A = \mathbf{R}$. The following results can be shown on similar lines.

(a) $S(X, A)$ is the maximal right (as well as left) quotient ring of B and is thus $Q_\sigma(B)$ (cf. [23], Theorem 8).

(b) The right and left classical rings of quotients of B coincide with the subring of $S(X, A)$ consisting of functions f_o whose domain U_o contains a dense cozero set.

(c) If X is a perfectly normal space then the maximal quotient ring and the classical quotient rings of $C(X, A)$ coincide. In particular, it is so, if X is a metric space.

(d) Let X be the space of ordinals less than or equal to the first uncountable ordinal Ω with the order topology. Then the set U of non-limit ordinals in X together with zero is a dense open set in X. Further, $Q(C(X, A)) = A^U$. Also, $Q_{cl}(C(X, A)) = \{k \in A^U :$ there exists $\alpha \in U$ with $k(x) = k(\alpha)$, for all $x \geq \alpha\} \neq A^U$. Indeed, for each nonzero x in U, $x = y + n$, for a unique limit ordinal y and a natural number n. We set $f(x) = n$. We take $f(0) = 1$. The function f so defined is in $C(U)$ and cannot respresent a classical fraction. To see this directly we first note that if g and h are in $R = C(X, A)$ and g is not a zerodivisor then there is an α in X with $\alpha < \Omega$ such that $g(x) = g(\alpha) \neq 0$ and $h(x) = h(\alpha)$ for $x \geq \alpha$. Thus $hg^{-1}(x)$ is the constant $h(\alpha)g(\alpha)^{-1}$ on $W = \{x \in X : x > \alpha\}$. Since $U \cap W$ contains isolated points of the type $\beta + n$ with $\beta = \sup\{\alpha + m : m \in \mathbf{N}\}$ and $n \in \mathbf{N}$, f and hg^{-1} cannot be equal on any dense open set.

(ii) If A has a smallest dense right ideal D, then by 2.1(g), $Q_{b,r}(A)$ is same as $Hom_{b,A}(D, D)$. The same is true if A is prime with nonzero socle, in view of 2.2 (b). So, Theorem 3.3 can be thought of as a topological version of one-half of Theorem 1.2 and Theorem 3.5 of the other. Further, this occurs when A is a C^*-algebra of operators on a Hilbert space H to itself containing all compact operators, in particular, $B(H)$; and for such $A's$ $Q_{b,r}(A) = B(H)$.

(iii) Let H be a Hilbert space and H' be the space of continuous linear functionals on H. As described by several authors including Y.Utumi, R.E.Johnson and J. Lambek (cf. [22],[14],[16],[17]),(see also 2.2 (b)), $Q_r(B(H))$ is the full linear ring $L(H_{\mathbf{F}})$, and $Q_l(B(H))$ is the full linear ring $L({}_{\mathbf{F}}H')$. By the Riesz Representation Theorem, ${}_{\mathbf{R}}H'$ is isomorphic to $H_{\mathbf{R}}$ and ${}_{\mathbf{C}}H'$ is conjugate linear isomorphic to $H_{\mathbf{C}}$. Consequently, $L({}_{\mathbf{F}}H')$ is anti-isomorphic to $L(H_{\mathbf{F}})$. Furthermore, in this case the maximal right and maximal left quotient rings of $R = B(H)$ are anti-isomorphic. We illustrate it in a diagram. Here V^a denotes the space of linear functionals on a vector space V.

$$\begin{array}{ccc}
L({}_{\mathbf{F}}H'^{aa}) \simeq Q_l(Q_r(Q_l(R))) & \overset{\simeq}{anti} & Q_r(Q_l(Q_r(R))) \simeq L(H_{\mathbf{F}}^{aa}) \\
| & & | \\
L(H'^{a}_{\mathbf{F}}) \simeq Q_r(Q_l(R)) & \overset{\simeq}{anti} & Q_l(Q_r(R)) \simeq L({}_{\mathbf{F}}H^a) \\
| & & | \\
L({}_{\mathbf{F}}H') \simeq Q_l(R) & \overset{\simeq}{anti} & Q_r(R) \simeq L(H_{\mathbf{F}}) \\
/ & & \backslash \\
R & = & B(H)
\end{array}$$

The theorem which follows is a neat, though not the most general, combined extension of Theorem 1.1., Theorem 1.2 and item 2.11 (h) (reformulated from [1], [17] and [9] respectively). The results apply to many topological spaces X, in particular, to the space of ordinals in Remarks 3.9 (i)(d) and each compactification of a discrete space.

Theorem 3.10. *Let Y be the set of isolated points of X. If Y is dense in X then*

$$Q_r(C_b(X, A)) = Q_r(C(X, A)) = Q_r(A)^Y \quad \textit{and}$$
$$Q_{b,r}(C_b(X, A)) = \mathcal{B}_b(Y, Q_{b,r}(A)),$$

the ring of bounded functions on Y to $Q_{b,r}(A)$.

Proof. We first note that in this case Y is the smallest member of $\mathcal{U}$ and thus, for any topological space Z and $U \in \mathcal{U}$, as done in 2.10, $C(U, Z)$ can be identified with a subset of $C(Y, Z) = Z^Y$. Let E be a dense right ideal of B_b. By Theorem 3.2(c) we have, for $x \in Y$, E_x is a dense right ideal of A. So, as in Theorem 3.3, for a $\phi \in Hom_{b,B_b}(E, B_b)$, the corresponding f represents a bounded function on Y to $Q_{b,r}(A)$. For any $f \in S(X, A)$ we choose a representative $f_1 \in \prod_{x \in Y} Hom_B(E^x, A)$ with E^x a dense right ideal in A. Let $E_{f_1} = \{g \in B_b : g$ vanishes outside a finite subset F of Y and for $x \in F,\ g(x) \in \cap_{y \in F} E^y\}$. It is a dense right ideal of B_b with $f_1 E_{f_1} \subset B_b$. □

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