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# When cyclic singular modules over a simple ring are injective

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**Abstract**

It is shown that a simple ring $R$ is Morita equivalent to a right PCI domain if and only if every cyclic singular right $R$-module is quasicontinuous.



## 1. Introduction

In this paper, a continuation of our earlier work [10], all rings are associative and have an identity. In addition, all modules are unitary. A cyclic module $X_R$ that is not isomorphic to $R_R$ is said to be a proper cyclic module. A ring $R$ is said to be a right PCI ring if every proper cyclic right $R$-module is injective. It has been shown [4,6] that, other than the semisimple artinian rings, right PCI rings are precisely the right noetherian, right hereditary domains for which every singular right $R$-module is injective. As semisimple artinian rings are well understood, the emphasis of PCI research rests on PCI domains. An example of a right PCI domain, not a division ring, is given in [3]. Rings for which every singular right module is injective are called right SI rings. The right PCI and right SI conditions are equivalent for domains. Furthermore, every right SI ring is a finite direct product of rings; one of these rings has essential socle and the others are Morita equivalent to SI domains (i.e. to PCI domains) (cf. [9]). Therefore, a non-artinian simple right SI ring is precisely a ring that is Morita equivalent to a right PCI domain.

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In [10, Theorem B], it was shown that a simple ring $R$ is right PCI if and only if every proper cyclic right $R$-module is quasiinjective. In this note we provide a stronger result, but the proof is simpler than that presented in [10]. Precisely, we prove the following

**Theorem 1.** *A simple ring $R$ is Morita equivalent to a right PCI domain, if and only if every cyclic singular right $R$-module is quasicontinuous.*

We remark that Theorem 1 can be stated as follows: A simple ring $R$ is right SI iff every cyclic singular right $R$-module is quasicontinuous.

Let $M_2(D)$ be the $2 \times 2$ matrix ring over a right PCI domain $D$. Then it is clear that every proper cyclic right $M_2(D)$-module is quasicontinuous. However, the following corollary shows that for any right PCI domain $D$, other than division ring, not every proper cyclic right module over $M_3(D)$ is quasicontinuous.

**Corollary 2.** *Let $R$ be a simple ring. If every proper cyclic right $R$-module is quasicontinuous, then $R$ is Morita equivalent to a right PCI domain and has right uniform dimension at most 2.*

The ring $M_2(D)$ shows that a ring of Corollary 2 is not necessarily right uniform, in general. But whenever it is right uniform, it must be a right PCI domain.

As a further application of Theorem 1 we obtain the following

**Theorem 3.** *For a right $V$-ring $R$ with $\mathrm{Soc}(R_R) = 0$ the following conditions are equivalent:*

(a) *Every cyclic singular right $R$-module is quasicontinuous.*
(b) *$R$ has a ring-direct decomposition $R = R_1 \oplus \cdots \oplus R_n$, where each $R_i$ is Morita equivalent to a right PCI domain.*

Theorem 3 is not true if we remove the condition that $R$ is a right $V$-ring. Take, for example, the ring of integers.

## 2. The proofs

We adopt the notations used in [10]. A module $M$ is called *continuous* if

(i) every submodule of $M$ is essential in a direct summand of $M$ and
(ii) a submodule isomorphic to a direct summand of $M$ is itself a direct summand of $M$.

A module $M$ is defined to be *quasicontinuous*, if $M$ satisfies (i) and for any direct summands $A$ and $B$ of $M$ with $A \cap B = 0$, $A \oplus B$ is also a direct summand of $M$. A module, that satisfies (i) only, is called a CS module. We have the following implications:

$$\text{injective} \Rightarrow \text{quasiinjective} \Rightarrow \text{continuous} \Rightarrow \text{quasicontinuous} \Rightarrow \text{CS}.$$

However, in general, these classes of modules are different. For basic properties of these modules we refer to [5,6,11,12].

**Proof of Theorem 1.** One direction is clear, since every ring which is Morita-equivalent to a right PCI domain has the property that all of its singular right modules are injective (cf. [9, Theorem 3.11]).

Conversely, let $R$ be a simple ring whose cyclic singular right $R$-modules are quasicontinuous. Then by [10, Theorem A], $R$ is right noetherian. If $\mathrm{Soc}(R_R) \neq 0$, then $R = \mathrm{Soc}(R_R)$, and hence $R$ is a simple artinian ring. We are done. Next, consider the case $\mathrm{Soc}(R_R) = 0$. We show that any artinian right $R$-module $A$ is semisimple:

Assume that $A \neq 0$, it is clear that $A_R$ is singular. Let $X$ be a cyclic submodule of $A$. Using [10, Lemma 3.1] we can show that $X \oplus \mathrm{Soc}(X)$ is cyclic. Hence $X \oplus \mathrm{Soc}(X)$ is quasicontinuous. Thus $\mathrm{Soc}(X)$ is $X$-injective, and so $\mathrm{Soc}(X)$ splits in $X$. This implies that $X = \mathrm{Soc}(X) \subset \mathrm{Soc}(A)$. This shows that $A$ is semisimple, as claimed.

Now, we prove that every singular cyclic module over $R$ is semisimple, or equivalently, for each essential right ideal $C$ of $R$, $R/C$ is semisimple. By the above claim, it suffices to that show $R/C$ is artinian. Hence $R$ is Morita equivalent to a right SI domain by [9, Theorem 3.11]. As right SI domains are the same as right PCI domains, the proof will be complete.

Assume on the contrary that there is an essential right ideal $A$ of $R$ such that $R/A$ is not artinian. Since $R$ is right noetherian, there exists an essential right ideal $L$ of $R$ which is maximal with respect to the condition that $M = R/L$ is not artinian. It follows that $M$ is uniform and $\mathrm{Soc}(M) = 0$. Moreover, for any nonzero submodule $N$ of $M$, $M/N$ is semisimple. Let $U$ and $V$ be submodules of $M$ with $0 \neq U \subset V \subset M$ and $U \neq V \neq M$. Then $V/U$ is a direct sum of finitely many simple modules. Consider the module $Q = M \oplus V$. Since $M$ is cyclic and $Q/(0, U) \cong M \oplus (V/U)$, we can use [10, Lemma 3.1] to show, by induction on the number of simple direct summands of $V/U$, that $Q/(0, U)$ is cyclic. Let $x \in Q$ such that $[xR + (0, U)]/(0, U) = Q/(0, U)$. Then $xR$ is not uniform. We can choose $x$ such that $xR$ contains $(M, 0)$. Hence $xR = M \oplus W$ where $(0, W) = xR \cap (0, V) \neq (0, 0)$. Since $xR$ is quasicontinuous, $W$ is $M$-injective. Therefore $W$ splits in $M$, a contradiction. □

**Proof of Corollary 2.** Let $R$ be a simple ring such that every proper cyclic right $R$-module is quasicontinuous. Then, in particular, every cyclic singular right $R$-module is quasicontinuous. By Theorem 1, $R$ is Morita equivalent to a right PCI domain. Hence, it is enough to show that the uniform dimension of $R_R$ is at most 2. (If the right uniform dimension of $R$ is 1, i.e. $R_R$ is uniform, then $R$ is a right PCI domain.)

We need only consider the case $\mathrm{Soc}(R_R) = 0$. Assume that the uniform dimension of $R_R$ is at least 3. Note that all uniform right ideals of $R$ are subisomorphic to each other. Let $U = U_1 \oplus \cdots \oplus U_n$ be an essential right ideal of $R$ where each $U_i$ is uniform and $n \geqslant 3$, and each $U_i$ embeds in $U_j$ for all $i, j$.

Let $U_1^*$ be the closure of $U_1$ in $R_R$. Then $R/U_1^*$ has uniform dimension $n-1$ (see, e.g., [5, 5.10(1)]). Clearly, $R/U_1^*$ contains a copy of $U_2 \oplus \cdots \oplus U_n$ which is therefore essential in $R/U_1^*$. Since, by hypothesis, $R/U_1^*$ is quasicontinuous, $R/U_1 = U_2^* \oplus \cdots \oplus U_n^*$ where each $U_i^*$ $(2 \leqslant i \leqslant n)$, is uniform, and for $i \neq j$, $U_i^*$ is $U_j^*$-injective. We may assume that each $U_i^*$ contains a copy of $U_i$. This implies that $U_i^*$ embeds in $U_j^*$, and hence each $U_i^*$ $(2 \leqslant i \leqslant n)$ is quasiinjective. Since $R$ is Morita equivalent to a right PCI domain, every quasiinjective right $R$-module is injective (see [8]). This shows that each $U_i^*$ $(i \geqslant 2)$, is a cyclic injective right $R$-module, and it is isomorphic to the injective hull of $U_i$. It implies that the injective hull of $R_R$ is finitely generated. Hence $R$ is right artinian by [2, Corollary 1.29]. This contradiction shows that the right uniform dimension of $R$ is at most 2, as desired. □

**Proof of Theorem 3.** (b) ⇒ (a) is clear, because every ring in (b) has the property that every singular right $R$-module is injective.

(a) ⇒ (b). Let $E$ be an essential right ideal of $R$, and set $M = R/E$. Then $M_R$ is a singular module, hence every cyclic submodule of any homomorphic image of $M$ is quasicontinuous. By a theorem of Osofsky and Smith (see [5, 7.13]), each factor module of $M$ has finite uniform dimension. By assumption, $R$ is a right $V$-ring. Hence $M$ is a $V$-module. Then it follows from [5, 12(1)], that $M$ is noetherian. Hence, by [5, 5.15(1)], $R/\operatorname{Soc}(R_R)$ is noetherian. But $\operatorname{Soc}(R_R) = 0$, hence $R$ is right noetherian. As $R$ is a right $V$-ring, it follows from [8, Theorem 2], that $R$ has the ring-direct decomposition $R = R_1 \oplus \cdots \oplus R_n$ where each $R_i$ is a simple ring. By Theorem 1, each $R_i$ is Morita equivalent to a right PCI domain. □

In [10, Theorem A], it was shown that a simple ring $R$ is right noetherian, if every cyclic singular right $R$-module is CS. However, the structure of these rings is still unknown.

**Problem.** Describe the structure of simple rings whose cyclic singular right modules are CS.

Finally, we remark that using the techniques presented in the proof of [10, Theorem B] an improvement of [10, Theorem B] has been obtained in [1], where it was shown that a simple ring $R$ is right PCI if and only if every proper cyclic right $R$-module is continuous. This result is obviously a direct consequence of Theorem 1 or Corollary 2.

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