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Euclidean pairs and quasi-Euclidean rings

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ABSTRACT

We study the interplay between the classes of right quasi-Euclidean rings and right K-Hermite rings, and relate them to projective-free rings and Cohn's GE_2-rings using the method of noncommutative Euclidean divisions and matrix factorizations into idempotents. Right quasi-Euclidean rings are closed under matrix extensions, and a left quasi-Euclidean ring is right quasi-Euclidean if and only if it is right Bézout. Singular matrices over left and right quasi-Euclidean domains are shown to be products of idempotent matrices, generalizing an earlier result of Laffey for singular matrices over commutative Euclidean domains.

1. Introduction and definitions

Inspired by Howie's work [11] on idempotents in transformation semigroups of sets, J.A. Erdos [6] proved that singular matrices over a field can be decomposed as products of idempotent matrices. This was extended in different directions by several authors (e.g. [1,3,13,7,21]). The decomposition of 2×2 matrices with a zero row over a commutative Euclidean domain was one of the main steps in Laffey's work [13]. In connection with

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this decomposition, we exploit the general notions of **right Euclidean pairs** and **right quasi-Euclidean rings** in this paper. This leads, in Section 2, to an easy and elementary proof for the idempotent decomposition of $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ over a possibly noncommutative right quasi-Euclidean ring. After proving this result, we show in Sections 2–3 that right Bézout rings of stable range one are right quasi-Euclidean, and that a general ring is right quasi-Euclidean if and only if it is a right K-Hermite ring (right Hermite ring in the sense of I. Kaplansky [12]) and a GE_2-ring (in the sense of P.M. Cohn [4]). In Section 3, it is also proved that matrix rings over right quasi-Euclidean rings remain right quasi-Euclidean. (Thus, for instance, $\mathbb{M}_n(\mathbb{Z})$ and $\mathbb{M}_n(\mathbb{Q}[x])$ provide new noncommutative examples of right and left quasi-Euclidean rings.) However, an example of Bergman in Section 4 shows that left and right quasi-Euclidean regular rings need not be Dedekind-finite. In Section 5, we revisit the theme of idempotent factorization of matrices, and prove our last main result (Theorem 25) that, over any left and right quasi-Euclidean domain, singular matrices are products of idempotent matrices.

A (not necessarily commutative) integral domain R is called a *right Euclidean domain* if there is a map $\varphi : R \setminus \{0\} \to \{0, 1, 2, \ldots\}$ such that, for any $a, b \in R$ with $b \neq 0$, there exists an equation $a = bq + r$ in R where either $r = 0$, or $\varphi(r) < \varphi(b)$. A *right chain ring* is a ring whose right ideals form a chain under inclusion; or equivalently, for any $a, b \in R$, we have either $aR \subseteq bR$ or $bR \subseteq aR$. A ring R is called a right Bézout ring if each finitely generated right ideal is principal. (For instance, any right chain ring and any principal right ideal ring is right Bézout.) A ring R is called *right K-Hermite* (after Kaplansky [12], but following the terminological convention of [16, I.4.23]) if for every pair $(a, b) \in R^2$ there exist an element $r \in R$ and a matrix $Q \in \mathrm{GL}_2(R)$ such that $(a, b) = (r, 0)Q$. Kaplansky has shown in [12] that any right K-Hermite ring is right Bézout, and Amitsur has shown in [2] that the converse holds if the ring in question is an integral domain. Needless to say, similar definitions and remarks can be made when the adjective "right" is replaced by "left".

An ordered pair (a, b) over any ring R is said to be a *right Euclidean pair* if there exist elements $(q_1, r_1), \ldots, (q_{n+1}, r_{n+1}) \in R^2$ (for some $n \geqslant 0$) such that $a = bq_1 + r_1$, $b = r_1 q_2 + r_2$, and

$$r_{i-1} = r_i q_{i+1} + r_{i+1} \quad \text{for } 1 < i \leqslant n, \text{ with } r_{n+1} = 0. \tag{$*$}$$

The notion of a *left Euclidean pair* is defined similarly. In the following, when we talk about "Euclidean pairs", we shall always mean *right* Euclidean pairs. If all pairs $(a, b) \in R^2$ are right Euclidean, we say that R is a *right quasi-Euclidean ring*. Clearly, factor rings and finite direct products of right quasi-Euclidean rings remain right quasi-Euclidean. For instance, right chain rings and factor rings of right Euclidean domains are right quasi-Euclidean rings. The notion of (right) quasi-Euclidean rings was introduced without a name by O'Meara [20], and with a name (and a somewhat different but equivalent definition) by Leutbecher [17]. More recently, Glivický and Šaroch [8] studied certain special classes of commutative quasi-Euclidean domains. In both [17] and [8], the authors gave examples of commutative domains that are quasi-Euclidean but not Euclidean. In Section 4, we show, however, that a left quasi-Euclidean domain need not be right quasi-Euclidean, but *it will be right quasi-Euclidean if and only if it is a right Ore domain*.

A *regular ring* is a ring R in which every element a is regular in the sense of von Neumann; that is, $a = ara$ for some $r \in R$. If r can be chosen to be a *unit* for every $a \in R$ (or equivalently, every $a \in R$ is the product of an idempotent and a unit), R is said to be a *unit-regular ring*. Another highly relevant notion needed for this paper is that of a GE_n-*ring*. Following P.M. Cohn, we let $\mathrm{E}_n(R)$ denote the group generated by the $n \times n$ elementary matrices over R, and let $\mathrm{GE}_n(R)$ denote the subgroup of $\mathrm{GL}_n(R)$ generated by $\mathrm{E}_n(R)$ and the group of invertible diagonal matrices. If $\mathrm{GL}_n(R) = \mathrm{GE}_n(R)$, we say that R is a GE_n-ring. For more detailed discussions on such rings, see [4] and [5].

For any matrix A in an $n \times n$ matrix ring $\mathbb{M}_n(R)$, we denote by $l.ann(A)$ $(r.ann(A))$ the left (right) annihilator of A in $\mathbb{M}_n(R)$. If both annihilators are nonzero, we say that A is *singular*. We shall sometimes use (without mention) the convenient fact that $r.ann(A) \neq 0$ if and only $Av = 0$ for some nonzero column vector $v \in R^n$. (Of course, a similar fact holds for $l.ann(A)$.) For other notations and ring-theoretic terminology not defined here, see [14,15].

2. Euclidean pairs and right quasi-Euclidean rings

The point of introducing the notion of right Euclidean pairs (rather than just the notion of right quasi-Euclidean ring) is that we can broadly study such pairs in any given noncommutative ring R, and try to relate their behavior to the ideal theory of R. For illustration, we begin with some simple examples of Euclidean pairs.

Example 1.

(1) For a, b, q in any ring R, both (bq, b) and $(a, 0)$ are Euclidean pairs as $bq = b \cdot q + 0$, and $a = 0 \cdot 1 + a$ along with $0 = a \cdot 0 + 0$. If b has a right inverse c, then (a, b) is a Euclidean pair for all $a \in R$ since $a = b(ca) + 0$.

(2) If (a, b) is a Euclidean pair and $q \in R$, we see easily that (b, a), $(a + bq, b)$, and $(b + aq, a)$ are also Euclidean pairs. From (1) and (2), it follows that, over a right chain ring R, all pairs in R^2 are Euclidean, so R is a right quasi-Euclidean ring.

(3) Let $s \in R$. If (a, b) is a Euclidean pair, then clearly (sa, sb) is a Euclidean pair. *The converse holds if s is not a left zero-divisor in R.* To see this, note that for any right Euclidean algorithm for (sa, sb) (as in $(*)$ in the Introduction), the "remainder terms" r_i are, by induction, all left divisible by s. Thus, left cancellation of the factor s from all equations in $(*)$ will give a right Euclidean algorithm for the original pair (a, b).

(4) Let (a, b) be a Euclidean pair. Then $(u^{-1}au, u^{-1}bu)$ is a Euclidean pair for every unit $u \in R$. More generally, if v, w are any right-invertible elements, then (av, bw) is a Euclidean pair. (The easy proofs are left to the reader.)
(5) If $a, b \in R$ are such that $a + bq$ is right-invertible for some q, then (a, b) is a Euclidean pair. In particular, if R is any ring of stable range one (in the sense of [14, Def. (20.10)]), then every pair (a, b) with $aR + bR = R$ is Euclidean.

This section is devoted to the study of 2×2 matrices, with an eye to the connections between the factorization of such matrices (into idempotents) and the notions of right quasi-Euclidean rings. In the balance of this paper, we shall use P.M. Cohn's convenient notation $P(q) := \begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(R)$, which he introduced in [5, p. 147]. Our first result is the following somewhat tricky lemma on expressing 2×2 matrices (over any ring) as products of idempotents.

Lemma 2. *Let $r, q_1, \ldots, q_n$ be elements in a ring R, and let E be a product of k idempotents (in $\mathbb{M}_2(R)$). Then $X = \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} E \cdot P(q_n) \cdots P(q_1)$ is a product of $k+n+1$ idempotents.*

Proof. Let $E' := P(q_n)^{-1} E P(q_n)$, which is a product of k idempotents. In terms of E', we have

$$X = \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} P(q_n) E' P(q_{n-1}) \cdots P(q_1)$$
$$= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q_n + r & 1 \end{pmatrix} E' P(q_{n-1}) \cdots P(q_1).$$

Since $E'' := \begin{pmatrix} 0 & 0 \\ q_n+r & 1 \end{pmatrix} E'$ is a product of $k + 1$ idempotents, the proof proceeds by induction on n. $\square$

Corollary 3. *Let R be a ring and let $a, b \in R$ be such that $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of k idempotents (in $\mathbb{M}_2(R)$). Then for any $q \in R$, $\begin{pmatrix} aq+b & a \\ 0 & 0 \end{pmatrix}$ is a product of $k+2$ idempotents. In particular, $\begin{pmatrix} b & a \\ 0 & 0 \end{pmatrix}$ is a product of $k + 2$ idempotents.*

Proof. In view of the factorization

$$\begin{pmatrix} aq + b & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} P(q),$$

the desired conclusion follows by applying Lemma 2 (with $n = 1$). $\square$

Next, we recall the following useful definition of P.M. Cohn [5, p. 148].

Definition 4. Let $X = \{x_1, x_2, \ldots\}$ be a countable set of noncommuting variables, and let $\mathbb{Z}\langle X\rangle$ be the free $\mathbb{Z}$-algebra generated by X. We define the n-th *right continuant polynomials*

$$p_n(x_1, \ldots, x_n) \in \mathbb{Z}\langle x_1, \ldots, x_n\rangle \subseteq \mathbb{Z}\langle X\rangle$$

by $p_0 \equiv 1$, $p_1(x_1) = x_1$, and inductively for $i \geqslant 2$ by

$$p_i(x_1, \ldots, x_i) = p_{i-1}(x_1, \ldots, x_{i-1})x_i + p_{i-2}(x_1, \ldots, x_{i-2}).$$

Thus, $p_2(x_1, x_2) = x_1x_2 + 1$, $p_3(x_1, x_2, x_3) = x_1x_2x_3 + x_3 + x_1$, etc.

We offer now the following characterization result for (right) Euclidean pairs.

Proposition 5. *Let a, b be elements in a ring R. The following are equivalent*:

(1) *(a, b) is a Euclidean pair.*
(2) *For some $n \geqslant 0$ there exist $q_1, \ldots, q_{n+1} \in R$ and $r_n \in R$ such that*

$$(a, b) = (r_n, 0)P(q_{n+1}) \cdots P(q_1).$$

(3) *For some $n \geqslant 0$ there exist $q_1, \ldots, q_{n+1} \in R$ and $r_n \in R$ such that $a = r_np_{n+1}(q_{n+1}, \ldots, q_1)$ and $b = r_np_n(q_{n+1}, \ldots, q_2)$.*

The equivalence of the statements (1) and (2) above shows, in particular, that every right quasi-Euclidean ring is right K-Hermite.

Proof. (1) $\Rightarrow$ (2). As in the definition of a Euclidean pair, we have $a = bq_1 + r_1$, $b = r_1q_2 + r_2, \ldots, r_{i-1} = r_iq_{i+1} + r_{i+1}, \ldots$ and $r_{n-1} = r_nq_{n+1}$ (for some $n \geqslant 0$). For technical convenience, we shall adopt (from here on) the convention that $r_0 := b$. If $n = 0$, the desired equation in (2) clearly holds. In general, we have

$$\begin{aligned}(a, b) &= (b, r_1)P(q_1) = (r_1, r_2)P(q_2)P(q_1) = \cdots \\ &= (r_n, 0)P(q_{n+1})P(q_n) \cdots P(q_1).\end{aligned}$$

Reversing the above steps (and introducing the elements $r_{n-1}, \ldots, r_1$ along the way, ending with $a = bq_1 + r_1$) shows that (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). For this, we use the elements q_i, r_i (with $r_{n+1} = 0$) in the "division process" associated with the Euclidean pair (a, b), and prove by "backward induction" on i that $r_i = r_np_{n-i}(q_{n+1}, \ldots, q_{i+2})$. (For $i = 0$ and $i = -1$, this will give the conclusions in (3), with the convention that $r_0 = b$ and $r_{-1} = a$.) Indeed, if the formula is known for the subscripts i and $i + 1$, then

$$\begin{aligned} r_{i-1} &= r_i q_{i+1} + r_{i+1} \\ &= r_n \big[p_{n-i}(q_{n+1}, \ldots, q_{i+2}) q_{i+1} + p_{n-i-1}(q_{n+1}, \ldots, q_{i+3}) \big] \\ &= r_n p_{n-i+1}(q_{n+1}, \ldots, q_{i+1}), \end{aligned}$$

which gives the desired formula for the subscript $i-1$. The converse (3) ⇒ (1) is proved in the same spirit, by taking the q_i's and r_n as given in (3), and defining the r_i's (for $i < n$) by the formula $r_i = r_n p_{n-i}(q_{n+1}, \ldots, q_{i+2})$. The details are left to the reader. □

Theorem 6. *Let $(a, b) \in R^2$ be a Euclidean pair. Then*

(a) *$aR + bR = r_n R$ where r_n is the element of R obtained in the proof of* (1) ⇒ (2) *in* Proposition 5.
(b) *If r_n is either central or not a left zero-divisor in R, then $aR \cap bR$ is also principal.*
(c) *$\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ is a product of $n+2$ idempotents in $\mathbb{M}_2(R)$, where n is the integer that appears in the definition of a Euclidean pair* (*cf. the Introduction*).

Proof. (a) is a well known conclusion in college algebra that is routinely proved by "working backwards" with the division formulas in the definition of a Euclidean pair given in the Introduction. From a more fancy matrix viewpoint, we can prove (a) as follows. Recall that in the proof of Proposition 5, $(a, b) = (r_n, 0)Q$ where $Q := P(q_{n+1}) \cdots P(q_1) \in \mathrm{GL}_2(R)$, so certainly $aR + bR \subseteq r_n R$. Writing $Q^{-1} = \left(\begin{smallmatrix} x & z \\ y & w \end{smallmatrix}\right)$, we have also $r_n = ax + by$, so $aR + bR = r_n R$. Finally, using (either one of) the assumptions on r_n in (b), a matrix argument in the proof of Kaplansky's [12, Lemma 3.3] shows that $aR \cap bR = azR$, where z is the $(1, 2)$-entry of Q^{-1}. (Note that, under the assumptions in (b), Kaplansky's proof works as long as we have $(a, b) = (d, 0)Q$ where $d \in R$ and $Q \in \mathrm{GL}_2(R)$. In other words, (a, b) need not be a Euclidean pair.)

(c) We can rewrite the matrix equation for (a, b) above in the alternative form:

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r_n & 0 \\ 0 & 0 \end{pmatrix} P(q_{n+1}) \cdots P(q_1) = \begin{pmatrix} 1 & r_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q_{n+1} & 1 \end{pmatrix} P(q_n) \cdots P(q_1).$$

Since $\left(\begin{smallmatrix} 0 & 0 \\ q_{n+1} & 1 \end{smallmatrix}\right)$ is an idempotent, Lemma 2 shows that $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ is a product of $n+2$ idempotents in $\mathbb{M}_2(R)$. □

Remark 7. (1) Statement (c) of the above Proposition was proved by Laffey in [13, Lemma 2] for matrices of the form $A = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ over a commutative Euclidean domain R using longer computational arguments (and without the bound $n+2$). For a concrete example, take $(a, b) = (14, 8)$ over $R = \mathbb{Z}$, for which $n = 2$, $q_1 = q_2 = 1$, $q_3 = 3$, and $r_2 = \gcd(14, 8) = 2$. Applying (c) above and thereafter the inductive proof of Lemma 2, we get the following factorization of A into $n+2 = 4$ idempotents:

$$\begin{pmatrix} 14 & 8 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 14 & 8 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}).$$

But of course, this factorization of A into idempotents is far from being unique. For instance, here is a shorter factorization (curiously with the same "last factor"):

$$\begin{pmatrix} 14 & 8 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 14 & 8 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}),$$

and it can be shown that this is in fact "a shortest" factorization for A.

(2) In the proof of Theorem 6, the element z is the $(1,2)$-entry of the matrix $Q^{-1} = P(q_1)^{-1} \cdots P(q_{n+1})^{-1}$. Since $P(q_i)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix}$, it is easy to see (e.g. by induction on n) that $z = (-1)^n p_n(q_2, \ldots, q_{n+1})$, where $p_n(x_1, \ldots, x_n)$ is the n-th right continuant polynomial introduced in Definition 4. This expression of z is of interest, since $aR \cap bR = azR$ if r_n is either central or not a left zero-divisor.

(3) In Theorem 6, the statement (c) is only a necessary but *not a sufficient condition* for (a,b) to be a Euclidean pair. To see this, let $\theta = \sqrt{-5}$ and $R = \mathbb{Z}[\theta]$ be the full ring of algebraic integers in the number field $\mathbb{Q}[\theta]$. The Dedekind domain R has class number 2, and its class group is generated by the ideal $-2R + (\theta+1)R$ (see [15, Example 2.19D]). The matrix $E = \begin{pmatrix} -2 & \theta+1 \\ \theta-1 & 3 \end{pmatrix}$ over R has trace 1 and determinant 0, so $E^2 = E$. Thus, $A := \begin{pmatrix} -2 & \theta+1 \\ 0 & 0 \end{pmatrix}$ has a simple idempotent factorization $\operatorname{diag}(1,0)E$. However, the ideal $-2R + (\theta+1)R$ is (by choice) *not* a principal ideal. In particular, $(-2, \theta+1)$ is not a Euclidean pair over R, according to Theorem 6(a).

(4) If the pair (a,b) is *left* Euclidean instead, a similar decomposition into products of idempotents holds for the matrix $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. We leave the details to the reader.

Using an idea of Zabavsky in [23, Theorem 2] and [24, Proposition 6], we obtain the following very simple criterion for Euclidean pairs over rings of stable range 1. (For the notion of "stable range 1", see [14, Def. (20.10)].)

Theorem 8. *Let R be a ring of stable range 1. Then $(a,b) \in R^2$ is a Euclidean pair if and only if the right ideal $aR + bR$ is principal.*

Proof. The "only if" part is Theorem 6(a). For the "if" part, assume that $aR + bR = dR$ for some $d \in R$, and write $a = da_0$, $b = db_0$, and $d = ax + by$. Letting $c = 1 - a_0x - b_0y$, we have $dc = d - ax - by = 0$, and $a_0x + (b_0y + c) = 1$. Since R has stable range 1, there exists $t \in R$ such that $u := a_0 + (b_0y + c)t$ is a unit. Left-multiplication by d then yields $du = a + byt + dct = a + byt$. We have now $a = b(-yt) + du$ and $b = (du)(u^{-1}b_0)$, so (a,b) is a Euclidean pair. $\square$

Corollary 9.

(1) (*Cf.* [23, *Theorem* 2].) *If R is a right Bézout ring with stable range* 1 (*e.g. R can be any semilocal right Bézout ring*), *then R is right quasi-Euclidean.*

(2) *If R is a unit-regular ring, then all matrix rings $\mathbb{M}_n(R)$ are right* (*and left*) *quasi-Euclidean.*

Proof. (1) follows directly from Theorem 8. For (2), it suffices to handle the case $n = 1$, since matrix rings over unit-regular rings remain unit-regular by [9, (4.7)]. For $n = 1$, the desired conclusion follows from part (1) since von Neumann regular rings are right (and left) Bézout by [9, (1.1)], and unit-regular rings have stable range 1 by [9, (7.42)]. □

Of course, if R is a commutative semilocal PID, then R is in fact a Euclidean domain; see, e.g. [24, Corollary 3]. Indeed, a Euclidean norm $\varphi : R \setminus \{0\} \to \{0, 1, 2, \dots\}$ is given by defining $\varphi(a)$ to be the length of a prime factorization of $a \in R \setminus \{0\}$. However, Corollary 9 applies more generally to some classes of rings which may be noncommutative non-domains.

3. Relations with Cohn's GE-rings

In this section, we work out the connections between the class of right quasi-Euclidean rings and the class of GE-rings introduced by P.M. Cohn [4] in 1966. To do this, we first recall some of the notations and terminology in Cohn's book [5]. For any $n \geqslant 1$, we denote by $\mathrm{GE}_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by the $n \times n$ elementary matrices and the invertible diagonal matrices. We say that R is a **GE_n-ring** if $\mathrm{GL}_n(R) = \mathrm{GE}_n(R)$. (Note that $\mathrm{GL}_1(R)$ and $\mathrm{GE}_1(R)$ are both the group of units of R, so R is always a GE_1-ring.) If R is a GE_n-ring for all $n \geqslant 2$, we say that R is a **GE-ring**. For instance, if R is a ring with stable range one, an easy induction argument on $n \geqslant 1$ shows that R is a GE_n-ring, so R gives a quick example of a GE-ring. This class of examples includes all semilocal rings and all unit-regular rings, according to [14, (20.9)] and [9, (7.42)] respectively.

For the next proposition, we'll be using again the important matrices $P(q) = \left(\begin{smallmatrix} q & 1 \\ 1 & 0 \end{smallmatrix}\right)$ introduced in Section 2. It is easy to see that $P(q) \in \mathrm{GE}_2(R)$ for all $q \in R$. Using a general factorization result of Cohn [5] for matrices in $\mathrm{GE}_2(R)$, we have the following important addition to Proposition 5 on the characterizations of (right) Euclidean pairs.

Proposition 10. *A pair $(a, b) \in R^2$ is a Euclidean pair if and only if $(a, b) = (r, 0)Q$ for some $r \in R$ and $Q \in \mathrm{GE}_2(R)$. In this case, Q can always be chosen to be in the group $\mathrm{E}_2(R)$ generated by the 2×2 elementary matrices over R.*

Proof. First assume that $(a, b) = (r, 0)Q$, where $r \in R$ and $Q \in \mathrm{GE}_2(R)$. By Cohn's factorization theorem in [5, p. 147], we can write the matrix $Q \in \mathrm{GE}_2(R)$ in the form $\mathrm{diag}(u, v)P(q_{n+1}) \cdots P(q_1)$ for suitable $q_1, \dots, q_{n+1} \in R$ and suitable units $u, v \in R$. Thus,

$$\begin{aligned}(a, b) &= (r, 0)\,\mathrm{diag}(u, v)P(q_{n+1}) \cdots P(q_1) \\ &= (ru, 0)P(q_{n+1}) \cdots P(q_1),\end{aligned}$$

so (2) ⇒ (1) in Proposition 5 shows that (a, b) is a Euclidean pair. Conversely, if (a, b) is a Euclidean pair, we can use a sequence of $\mathrm{E}_2(R)$-actions on the right of (a, b) to bring it to either $(*,0)$ or $(0,*)$. If $(*,0)$ is reached, we are done. If we reach some $(0,r)$ instead, we may further perform the elementary transformations: $(0,r) \mapsto (r,r) \mapsto (r,0)$ to reach the desired form $(r,0)$. $\square$

By applying the above Proposition to *all pairs* $(a,b) \in R^2$, we retrieve the following characterization theorem of Leutbecher [17] for right quasi-Euclidean rings. For the sake of completeness, we include a simple proof using a classical result of Kaplansky.

Theorem 11. *For any ring R, the following statements are equivalent*:

(A) *R is right quasi-Euclidean.*
(B) *R is a* GE*-ring that is right K-Hermite.*
(C) *R is a* GE_2*-ring that is right K-Hermite.*
(D) *For any $a,b \in R$, $(a,b) = (r,0)Q$ for some $r \in R$ and $Q \in \mathrm{GE}_2(R)$.*
(E) *For any $a,b \in R$, $(a,b) = (r,0)Q$ for some $r \in R$ and $Q \in \mathrm{E}_2(R)$.*

Proof. (B) $\Rightarrow$ (C) $\Rightarrow$ (D) are tautologies, and (D) $\Rightarrow$ (B) follows from Kaplansky's result [12, Theorem 7.1]. Finally, the equivalence of (A), (D) and (E) is clear from Proposition 10. $\square$

While Theorem 11 was a relatively easy result, there is yet another characterization for the right quasi-Euclidean property in the case of *domains*, which we'll now present. Following [5], we say that a ring R is *projective-free* if every finitely generated right (equivalently left) projective module is free of a unique rank. In the case where such a ring R is a commutative domain, Bhaskara Rao showed in [3, Lemma 1] that, if $A = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(R)$ is a product of idempotent matrices, then the ideal $aR + bR$ is principal. [Thus, for instance, the matrix $\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(k[x,y])$ (for any commutative ring $k \neq 0$) cannot be a product of idempotent matrices.[1] The same remark can be made about the matrix $\left(\begin{smallmatrix} 2 & y \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{Z}[y])$.] The following lemma is a broad generalization of Bhaskara Rao's result to the noncommutative non-domain case, with a significantly simplified proof.

Lemma 12. *Let R be a projective-free ring and let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{M}_2(R)$. If $A = TE$ for some $T, E \in \mathbb{M}_2(R)$ where $E^2 = E \neq I_2$, then there exists $Q \in \mathrm{GL}_2(R)$ such that $(a,b) = (r,0)Q$ for some $r \in R$ (and hence $aR + bR = rR$). In particular, if all matrices $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ are products of idempotents in $\mathbb{M}_2(R)$, R must be a right K-Hermite ring.*

Proof. We may assume that $E \neq 0$, in which case the projective-free assumption on R implies that $E = Q^{-1}\operatorname{diag}(1,0)Q$ for some $Q \in \mathrm{GL}_2(R)$. Then $A = TQ^{-1}\operatorname{diag}(1,0)Q =$

[1] After factoring out a maximal ideal, we may assume that k is a field and apply the Quillen–Suslin Theorem [16, V.2.9] that $k[x,y]$ is projective-free.

$\left(\begin{smallmatrix} r & 0 \\ s & 0 \end{smallmatrix}\right) Q$, where $\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right)$ is the first column of TQ^{-1}. Thus, $(a, b) = (r, 0)Q$. The rest of the Lemma is now clear. □

Of course, in the Lemma above, the "projective-free" assumption on the ring R was essential, as we have seen from the example $R = \mathbb{Z}[\theta]$ ($\theta = \sqrt{-5}$) given in Remark 7(3). Over this non projective-free Dedekind domain, we have the idempotent factorization

$$\begin{pmatrix} -2 & \theta+1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & \theta+1 \\ \theta-1 & 3 \end{pmatrix}.$$

Here, the second idempotent factor on the RHS is *not* diagonalizable (over R), and the ideal $-2R + (\theta+1)R$ is non-principal.

Theorem 13. *A domain R is right quasi-Euclidean if and only if R is a projective-free GE_2-ring such that every matrix $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ is a product of idempotents in $\mathbb{M}_2(R)$.*

Proof. The "if" part follows from Lemma 12 and (A) ⇔ (C) in Theorem 11 (even without the domain assumption on R). For the "only if" part, consider any right quasi-Euclidean domain R. By Theorem 11, R is a right K-Hermite GE_2-ring, and by Theorem 6(c), every matrix $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ is a product of idempotents. It only remains to show that R is projective-free. Since R is right Bézout, it is a right semihereditary domain, so Albrecht's theorem [15, (2.29)] implies that every finitely generated projective right R-module P is isomorphic to $J_1 \oplus \cdots \oplus J_k$ where each J_i is a finitely generated (and hence principal) right ideal of R. Each J_i is free, so P is also free. □

Next, we'll give another application of Theorem 11 by proving that the "right quasi-Euclidean" property is preserved by matrix ring extensions. This result was absent from the papers [17] and [8] as Leutbecher, Glivický and Šaroch were primarily interested only in the *commutative* case of quasi-Euclidean rings.

Theorem 14.

(1) *If R is a GE_{kn}-ring, then $S = \mathbb{M}_k(R)$ is a GE_n-ring.*
(2) *If R is a GE-ring, so is $S = \mathbb{M}_k(R)$ for every $k \geqslant 1$.*
(3) *If R is a right quasi-Euclidean ring, so is $S = \mathbb{M}_k(R)$ for every $k \geqslant 1$.*

Proof. For (1), consider any matrix $A \in \mathrm{GL}_n(S)$. We may view A as an (invertible) $kn \times kn$ matrix over R. Assuming that $\mathrm{GL}_{kn}(R) = \mathrm{GE}_{kn}(R)$, A is a product of $kn \times kn$ elementary matrices and invertible diagonal matrices over R. Thus, we are done if we can show that any elementary matrix $B = (B_{ij}) \in \mathbb{M}_{kn}(R)$ (where $i, j \in [1, n]$ and each $B_{ij} \in S = \mathbb{M}_k(R)$) is in the group $\mathrm{GE}_n(S)$. Say $B = I_{kn} + xe_{pq}$ where $x \in R$, $p \neq q$, and e_{pq} is one of the matrix units in $\mathbb{M}_{kn}(R)$. If the non-diagonal entry x does not occur in any one of the B_{ii}'s, then B is an elementary matrix in $\mathbb{M}_n(S)$. Now assume that the entry x occurs in some B_{ii}. Then the diagonal block E_{ii} is *invertible*, and the blocks E_{jj} must be I_k for $j \neq i$. In this case, B is a "diagonal invertible matrix" in $\mathbb{M}_n(S)$, so it is in $\mathrm{GE}_n(S)$. We have thus proved that S is a GE_n-ring.

Clearly, (1) implies (2). For (3), assume R is right quasi-Euclidean. Then R is right K-Hermite (by Proposition 5), and hence so is $S = \mathbb{M}_k(R)$, according to another result of Kaplansky [12, Theorem 3.6]. Also, R is a GE-ring by Theorem 11, so by part (2) above, S is a GE-ring as well. Applying Theorem 11 again, we see that S is right quasi-Euclidean, as desired. □

To conclude this section, we'll prove one more result about the preservation of the quasi-Euclidean property. This result gives another natural criterion for a ring R to be right quasi-Euclidean — in terms of the right quasi-Euclideanness of R modulo its Jacobson radical $\mathrm{rad}(R)$ (or any ideal inside this radical).

Theorem 15. *For any ideal $I \subseteq \mathrm{rad}(R)$, R is right quasi-Euclidean if and only if R is right Bézout and R/I is right quasi-Euclidean.*

Proof. We need only prove the "if" part, so assume R is right Bézout with $\overline{R} := R/I$ right quasi-Euclidean. Theorem 11 implies that $\overline{R}$ is a GE_2-ring that is right K-Hermite. Also, if both of these properties can be lifted to R, Theorem 11 would, in turn, imply that R is right quasi-Euclidean. Now the GE_2-property certainly lifts to R since any $U \in \mathrm{GL}_2(R)$ can be "matched" modulo I with a product of matrices $U_i \in \mathrm{GL}_2(R)$ each with a unit entry (and any such U_i is easily seen to be in $\mathrm{GE}_2(R)$). Thus, it only remains to lift the K-Hermite property. Given any $(a, b) \in R^2$, we may first write $aR + bR = dR$ for some $d \in R$. Thus, there exist $a_0, b_0, x, y \in R$ such that $a = da_0$, $b = db_0$, and $d = ax + by$. For $c := 1 - a_0x - b_0y$, we have $dc = 0$ and $a_0R + b_0R + cR = R$. Since $\overline{R}$ is right K-Hermite, it has stable range $\leqslant 2$ by [19, Proposition 8], so R also has stable range $\leqslant 2$ (in view of $I \subseteq \mathrm{rad}(R)$). Therefore, there exist $p, q \in R$ such that $(a_0 + cp)R + (b_0 + cq)R = R$. As $\overline{R}$ is right K-Hermite, the right unimodular row $(\overline{a_0 + cp}, \overline{b_0 + cq})$ can be completed to an invertible matrix over $\overline{R}$ according to [19, p. 32]. Therefore, $(a_0 + cp, b_0 + cq)$ can also be completed to an invertible matrix $U = \begin{pmatrix} a_0+cp & b_0+cq \\ * & * \end{pmatrix}$ over R. Since $dc = 0$, it follows that

$$(d, 0)U = \big(d(a_0 + cp), d(b_0 + cq)\big) = (a, b),$$

so $(a, b)U^{-1} = (d, 0)$. This checks that R is right K-Hermite, as desired. □

4. Issues of left–right symmetry and Dedekind-finiteness

In the study of right Euclidean pairs and right quasi-Euclidean rings, the question of left–right symmetry comes up naturally. To begin our considerations on this issue, we first point out that, in general, a left Euclidean pair $(a, b) \in R^2$ need not be right Euclidean.

For instance, let R be the ring of 2×2 lower triangular matrices over a field k. If we consider the matrix units $e = E_{11}$ and $r = E_{21}$, then the relation $re = E_{21}E_{11} = E_{21} = r$ shows that (e, r) is a *left* Euclidean pair. But it is easy to see that the right ideal $eR + rR = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}$ is non-principal, so (e, r) is not a right Euclidean pair by Theorem 6(a). While this example was on the element level, we will show by another example below that, in general, left quasi-Euclidean rings also need not be right quasi-Euclidean — even in the case of domains.

Example 16. Let σ be a non-surjective endomorphism of a field k, and let R be the domain $k[x, \sigma]$ of twisted polynomials in x over k (defined by taking $xa = \sigma(a)x$ for all $a \in k$). It is well known that R is a left Euclidean domain with respect to the usual degree function; in particular, R is a left quasi-Euclidean domain. Now, take any element $a \in k \setminus \sigma(k)$. It is easy to see that $axR \cap xR = 0$, and that the right ideal direct sum $axR + xR$ is *non-principal*. Thus, R is not right Bézout, and hence not a right quasi-Euclidean domain. Indeed, while (ax, x) is (obviously) a left Euclidean pair, it is *not* a right Euclidean pair by Theorem 6. Also, note that R being a left PID implies that it is a projective-free ring; see, for instance, [15, Theorem 2.24]. Thus, by Lemma 12, the fact that $axR + xR$ is non-principal implies also that the matrix $A = \begin{pmatrix} ax & x \\ 0 & 0 \end{pmatrix}$ is not right-divisible by any idempotent matrix (other than I_2). In particular, A is not a product of idempotent matrices over R. We note, however, that *for any two elements* a, x in any ring, the “other” pair (xa, x) is always a right Euclidean pair (with the “division process” stopping in one step). According to Theorem 6(c), the matrix $B = \begin{pmatrix} xa & x \\ 0 & 0 \end{pmatrix}$ should be a product of two idempotents, and indeed, $B = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}$. As for the matrix $C = \begin{pmatrix} x & xa \\ 0 & 0 \end{pmatrix}$ (obtained by permuting the two columns of B), Corollary 3 guarantees that C is a product of four idempotents. For such an explicit factorization, see [1, Lemma 1(b)]. But curiously enough, C turns out to be a product of two idempotents too! We leave this as an exercise for the reader.

In the standard theory of right rings of quotients (see, e.g. [15, §10B]), an integral domain R is said to be *right Ore* if $rR \cap sR \neq 0$ for any nonzero $r, s \in R$. Intuitively speaking, what “went wrong” in Example 16 above was that the left Euclidean domain in question failed to be a right Ore domain. With such a “realization”, we can then try to exploit Theorem 11 to prove a result on “partial” left–right symmetry of the quasi-Euclidean property of rings. Part (A) of the following result is basically the same as Zabavsky’s [24, Proposition 9]. However, the proof given below for this part is shorter and more conceptual.

Theorem 17.

(A) *A left quasi-Euclidean ring R is right quasi-Euclidean if and only if it is right Bézout. In particular, a regular ring is left quasi-Euclidean if and only if it is right quasi-Euclidean.*

(B) *A left quasi-Euclidean domain R is right quasi-Euclidean if and only if it is a right Ore domain.*

Proof. (A) For the first statement, the "only if" part is true (without any "left" assumption on R) by Proposition 5(2), since right K-Hermite rings are right Bézout. Conversely, if R is left K-Hermite and right Bézout, then R is right K-Hermite by a result of Menal and Moncasi [19, Proposition 8(ii)]. Adding a GE_2-ring assumption on R and applying Theorem 11, we arrive at the "if" part of the first statement in (A). The second part of (A) follows trivially from the first part since any regular ring is left and right Bézout (by [9, Theorem 1.1]).

(B) According to [5, Prop. 2.3.17], the statement (B) is true *for domains* if "quasi-Euclidean" is replaced throughout by "Bézout". Thus, by Amitsur's result in [2] (see also [1, Theorem 16]), (B) is true (for domains) if "quasi-Euclidean" is replaced throughout by "K-Hermite". Therefore, again by adding the GE_2-ring assumption on R and applying Theorem 11, we arrive at the desired conclusion in (B). □

We complete this section by addressing the question of *Dedekind-finiteness* for quasi-Euclidean rings. Recall that a ring R is called Dedekind-finite if, for any $a, b \in R$, $ab = 1 \Rightarrow ba = 1$. This property was mentioned several times in Kaplansky's paper [12]. However, Kaplansky left open the question whether left or right K-Hermite rings would satisfy this finiteness property. The answer to this question turns out to be "no" — even for left and right quasi-Euclidean regular rings, according to the following example kindly communicated to us by G. Bergman.

Example 18. Let A be the power series ring $k[\![x]\!]$ over a field k, and let $K = k(\!(x)\!)$ be the Laurent series field, which is the quotient field of A. Instead of working with $\mathrm{End}_k(A)$ (the full ring of k-vector space endomorphisms of A), Bergman has introduced in [10, Example 1] the following celebrated subring:

(A) $R = \{f \in \mathrm{End}_k(A)\colon \exists f_0 \in K \text{ such that } (f - f_0)(x^n A) = 0 \text{ for some } n \geqslant 1\}$.

This definition makes sense since, by multiplication, any $f_0 \in K$ maps A linearly into K. Similarly, all elements of A map A into A, so we may view A as a subring of R. In [10] (where Bergman's example first appeared; see also [9, Example 4.26]), it has been shown that R is a regular ring that is not Dedekind-finite. It turns out that R *is a left and right quasi-Euclidean ring*. In view of Theorem 17, it suffices to show that R is *left* quasi-Euclidean. Since elements of R are vector space endomorphisms of A, we can speak of their kernels and images. We first prove the following crucial fact about right divisibility in the ring R:

(B) *For any $f, g \in R$, $f \in Rg$ if and only if* $\ker(g) \subseteq \ker(f)$.

It suffices to prove the "if" part, so assume $\ker(g) \subseteq \ker(f)$. Since R is a regular ring, we may assume that g is an *idempotent*. (If $g = ghg \in R$, simply replace g by the idempotent hg without changing Rg or $\ker(g)$.) Then $A = \mathrm{im}(g) \oplus \ker(g)$. As the equation $f(a) = fg(a)$ holds trivially for both $a \in \mathrm{im}(g)$ and $a \in \ker(g) \subseteq \ker(f)$, it holds for all $a \in A$. Thus, $f = fg \in Rg$.

We'll now prove that every $(f,g) \in R^2$ is a left Euclidean pair. Let $\varphi : R \to K$ be the ring homomorphism which sends any $f \in R$ to the (uniquely determined) Laurent series $f_0 \in K$ in the defining equation (A) above. Since φ is a homomorphism onto a field, a single elementary transformation will bring the pair (f,g) into one where one of the components is in $\ker(\varphi)$. Thus, we may assume, say, $\varphi(g) = 0$; in particular, $\dim_k \mathrm{im}(g) < \infty$. Now we'll add to g the endomorphism $x^n f$ where n is chosen large enough so that $x^n A \cap \mathrm{im}(g) = 0$. Then we claim that

(C) $\ker(g + x^n f) = \ker(f) \cap \ker(g)$.

It suffices to prove the inclusion "$\subseteq$". For any $a \in \ker(g + x^n f)$, we have $g(a) + x^n(f(a)) = 0$. Since $x^n A \cap \mathrm{im}(g) = 0$, this implies that $g(a) = f(a) = 0$, so $a \in \ker(f) \cap \ker(g)$. Having proved (C), we deduce from (B) that $f \in R(g + x^n f)$, so another elementary transformation brings $(f, g + x^n f)$ to $(0, g + x^n f)$, which proves our claim that every pair $(f,g) \in R^2$ is left Euclidean.

Remark 19. Since Bergman's ring R above is left (and right) quasi-Euclidean, it is also left (and right) K-Hermite, so it has stable range $\leqslant 2$ (and hence equal to 2) by a result of Menal and Moncasi [19, Proposition 8(i)]. Thus, R gives an example of a ring with stable range two that is not Dedekind-finite. The existence of such examples was noted earlier by Menal and Moncasi [19, Example 1] and Stepanov [22], but it is worth noting again that Bergman's example is a regular GE-ring with both left and right division algorithms for arbitrary pairs of elements.

5. Singular matrices over quasi-Euclidean domains

In this final section, we return to the theme of the factorization of singular matrices over right quasi-Euclidean rings. Before we treat the case of $n \times n$ matrices, we first prove the following result on 2×2 matrices which generalizes Laffey's Lemma 2 in [13] as well as Theorem 10 of Alahmadi, Jain and Leroy in [1], upon noting that it applies to both right Euclidean domains and right chain domains (which were the cases treated in [13] and [1]).

Theorem 20. *Let R be a right quasi-Euclidean domain and let $A \in \mathbb{M}_2(R)$ be such that $l.ann(A) \neq 0$. Then A is a product of idempotent matrices.*

Proof. Say $(x, y)A = 0$ where $(x, y) \neq (0, 0)$. By Theorem 6(a), $xR + yR \neq 0$ is a *principal* right ideal. Since R is a domain, we can find $\alpha, x', y' \in R$ such that $(x, y) = \alpha(x', y')$ and $(x', y')A = (0, 0)$ with $x'R + y'R = R$. Applying Proposition 5 to the Euclidean pair (x', y') (and noting that the "r_n" in the proof of Theorem 6 is now a unit), we see that (x', y') is the first row of an invertible matrix,[2] and therefore also the second row of an invertible matrix, say P. Then $(x', y')A = (0, 0)$ implies that PAP^{-1} has the form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$. By Theorem 6, PAP^{-1} and hence A are products of idempotent matrices in $\mathbb{M}_2(R)$, as desired. $\square$

Note that the "domain" assumption in this theorem cannot be removed. Indeed, if $R = R_1 \times R_2$ is left and right quasi-Euclidean where each R_i is a nonzero ring, then any matrix $A \in \mathbb{M}_2(R_1) \subseteq \mathbb{M}_2(R)$ is left (and right) annihilated by the identity matrix in $\mathbb{M}_2(R_2) \subseteq \mathbb{M}_2(R)$. But of course A may not be a product of idempotent matrices over R_1 (let alone over R). The same remark on the essentialness of the domain assumption applies also to most of our results in the rest of this section.

The generalization of Theorem 20 to matrices of arbitrary size (Theorem 25 below) can be obtained along the same lines as in Laffey's Theorem 2 in [13] and Theorem 22 of Alahmadi, Jain and Leroy in [1]. Note that in both of these references the matrix in question is assumed to be *singular*, in the sense that its left and right annihilators are both nonzero. In Proposition 22 below, we'll show that, over a right quasi-Euclidean domain R, a matrix $A \in \mathbb{M}_n(R)$ is already singular if $l.ann(A) \neq 0$. To see this, we first prove the following lemma which holds without a domain assumption on R.

Lemma 21. *Let R be a right quasi-Euclidean ring. Then*

(a) *For any $A \in \mathbb{M}_n(R)$, there exists an invertible matrix $P \in \mathbb{M}_n(R)$ such that AP is lower triangular.*
(b) *Let $T = (t_{ij}) \in \mathbb{M}_n(R)$ be a lower triangular matrix with at least one zero entry on the diagonal. Then $r.ann(T) \neq 0$.*

Proof. (a) follows from Proposition 5(2) and Kaplansky's result [12, Theorem 3.5]. The proof of (b) proceeds by total induction on n. The case $n = 1$ being trivial, we may assume that the conclusion is true for matrices of smaller size than $n \times n$. If $t_{11} \neq 0$, then $t_{i+1,i+1} = 0$ for some $i \geqslant 1$. In this case, the conclusion is at hand by applying the inductive hypothesis to the southeast $(n-i) \times (n-i)$ corner of A. Thus, we may assume that $t_{11} = 0$, in which case A has northwest 2×2 corner $\begin{pmatrix} 0 & 0 \\ t_{21} & t_{22} \end{pmatrix}$. Since (t_{21}, t_{22}) is a right Euclidean pair, Proposition 5(2) shows that there exist an invertible 2×2 matrix P and an element $r \in R$ such that $(t_{21}, t_{22})P = (r, 0)$, and hence $\begin{pmatrix} 0 & 0 \\ t_{21} & t_{22} \end{pmatrix} P = \left(\begin{smallmatrix} 0 & 0 \\ r & 0 \end{smallmatrix}\right)$. For the invertible block matrix $Q := \operatorname{diag}(P, I_{n-2})$, the product TQ remains lower triangular,

[2] As a cautionary note to the reader, we point out that, over a *noncommutative* ring R in general, not every right unimodular pair can be realized as a row of a matrix in $\mathrm{GL}_2(R)$.

with diagonal entries $0, 0, t_{33}, \dots, t_{nn}$. By the case we have already dealt with, $(TQ)v = 0$ for some column $v \neq 0$. Since $v \neq 0 \Rightarrow Qv \neq 0$, we have $r.ann(T) \neq 0$, as desired. □

Proposition 22. *Let R be a right quasi-Euclidean domain and $A \in \mathbb{M}_n(R)$. Then $l.ann(A) \neq 0$ implies that $r.ann(A) \neq 0$.*

Proof. Let $P \in \mathbb{M}_n(R)$ be an invertible matrix such that $AP = T$ is lower triangular. Let $wA = 0$ where w is a nonzero row vector in R^n. This implies that $wT = 0$. Since R is a *domain*, some element on the diagonal of T must be zero. Lemma 21(b) then shows that $r.ann(T) \neq 0$, and hence also $r.ann(A) \neq 0$. □

Remark 23. (A) Classically, the conclusion of Proposition 22 is well known to be true for matrices A over a *commutative* ring; see, e.g. McCoy's book [18, p. 161]. However, we'll see in (B) below that this conclusion is not true in the noncommutative case, even for 2×2 matrices over domains.

(B) We note that the *converse* of Proposition 22 is not true either. To give an example for this, it is convenient to "switch sides". The opposite ring version of Proposition 22 says that, over a *left* quasi-Euclidean ring R, $r.ann(A) \neq 0 \Rightarrow l.ann(A) \neq 0$ (for any $A \in \mathbb{M}_n(R)$). However, for the left (quasi) Euclidean domain $R = k[x, \sigma]$ in Example 16, the matrix $A = \left(\begin{smallmatrix} ax & x \\ 0 & 0 \end{smallmatrix}\right)$ (for any $a \in k \setminus \sigma(k)$) is left-annihilated by $\operatorname{diag}(0, 1)$, but $r.ann(A) = 0$ since (as we have pointed out earlier) $axR \cap xR = 0$.

We have nevertheless the following obvious consequence of Proposition 22.

Corollary 24. *Let R be a right and left quasi-Euclidean domain. For any $A \in \mathbb{M}_n(R)$, we have $r.ann(A) = 0$ if and only if $l.ann(A) = 0$.*

We can now state the final result in this paper.

Theorem 25. *Let R be a right and left quasi-Euclidean domain. Then every matrix $A \in \mathbb{M}_n(R)$ with $l.ann(A) \neq 0$ (equivalently, $r.ann(A) \neq 0$) is a product of idempotent matrices.*

With the 2×2 case already settled in Theorem 20, the proof follows along the same lines as in Laffey's Theorem 2 in [13] and Theorem 22 of Alahmadi, Jain and Leroy in [1]. We remark, however, that Bhaskara Rao has shown in [3] that the conclusion of Theorem 25 is not true even over commutative principal ideal domains.

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