FINITE GENERATION OF LIE ALGEBRAS ASSOCIATED TO ASSOCIATIVE ALGEBRAS

ADEL ALAHMEDI [A], HAMED ALSULAMI[A], S. K. JAIN [A,B] , EFIM ZELMANOV[A,C,1]

Abstract. Let F be a field of characteristic not 2 . An associative F-algebra R gives rise to the commutator Lie algebra $R^{(-)} = (R, [a, b] = ab - ba)$. If the algebra R is equipped with an involution $* : R \to R$ then the space of the skew-symmetric elements $K = \{a \in R \mid a^* = -a\}$ is a Lie subalgebra of $R^{(-)}$. In this paper we find sufficient conditions for the Lie algebras $[R, R]$ and $[K, K]$ to be finitely generated.

1. Introduction

Let F be a field of characteristic not 2. An associative F-algebra R gives rise to the commutator Lie algebra $R^{(-)} = (R, [a, b] = ab - ba)$ and the Jordan algebra $R^{(+)} = (R, a \circ b = \frac{1}{2}(ab + ba))$. If the algebra R is equipped with an involution $* : R \to R$ then the space of skew-symmetric elements $K = \{a \in R \mid a^* = -a\}$ is a Lie subalgebra of $R^{(-)}$, the space of symmetric elements $H = \{a \in R \mid a^* = a\}$ is a Jordan subalgebra of $R^{(+)}$. Following the result of J.M. Osborn (see[4]) on finite generation of the Jordan algebras $R^{(+)}, H$, I. Herstein [4] raised the question about finite generation of Lie algebras associated to R. In this paper we find sufficient conditions for the Lie algebras $[R^{(-)}, R^{(-)}], [K, K]$ to be finitely generated.

Theorem 1. *Let R be a finitely generated associative F-algebra with an idempotent e such that $ReR = R(1-e)R = R$. Then the Lie algebra $[R, R]$ is finitely generated.*

The following example shows that the idempotent condition can not be dropped.

Example 1. The algebra $R = \begin{pmatrix} F[x] & F[x] \\ 0 & F[x] \end{pmatrix}$ of triangular 2×2 matrices over the polynomial algebra $F[x]$ is finitely generated. However the Lie algebra

$[R, R] = \begin{pmatrix} 0 & F[x] \\ 0 & 0 \end{pmatrix}$ is not.

Key words and phrases. associative algebra, Lie subalgebra, finitely generated.

Theorem 2. *Let R be a finitely generated associative F-algebra with an involution $*: R \to R$. Suppose that R contains an idempotent e such that $ee^* = e^*e = 0$ and $ReR = R(1-e-e^*)R = R$. Then the Lie algebra $[K, K]$ is finitely generated.*

The following example shows that the condition on the idempotent cannot be relaxed.

Example 2. Consider the associative commutative algebra $A = F[x,\ y]/id(x^2)$ with the automorphism φ of order 2: $\varphi(x) = -x$, $\varphi(y) = y$. The algebra $R = M_2(A)$ of 2×2 matrices over A has an involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d^\varphi & b^\varphi \\ c^\varphi & a^\varphi \end{pmatrix}$. We have $[K, K] \leq xM_2(F[y])$, $dim_F[K, K] = \infty$, which implies that algebra $[K, K]$ is not finitely generated.

W.E.Baxter [2] showed that if R is a simple $F-$ algebra, which is not ≤ 16 dimensional over its center Z then the Lie algebra $[K, K]/[K, K] \bigcap Z$ is simple.

Theorem 3. *Let R be a simple finitely generated $F-$ algebra with an involution $*: R \to R$. Suppose that R contains an idempotent e such that $ee^* = e^*e = 0$. Then the Lie algebra $[K, K]/[K, K] \bigcap Z$ is finitely generated.*

2. Finite generation of Lie algebras $[R, R]$

Consider the Peirce decomposition $R = eRe + eR(1-e) + (1-e)Re + (1-e)R(1-e)$. The components $eR(1-e), (1-e)Re$ lie in $[R,\ R]$ since $eR(1-e) = [e, eR(1-e)]$, $(1-e)Re = [e, (1-e)Re]$.

Lemma 1. *The Lie algebra $[R,\ R]$ is generated by $eR(1-e) + (1-e)Re$.*

Proof. We only need to show that

$$[eRe, eRe] + [(1-e)R(1-e), (1-e)R(1-e)] \subseteq \text{ Lie } \langle eR(1-e), (1-e)Re \rangle .$$

From $R = R(1-e)R$ it follows that $eRe = eR(1-e)Re$. Hence an arbitrary element from eRe can be represented as a sum $\sum_i a_i b_i, a_i \in eR(1-e), b_i \in (1-e)Re$. Now $a_i b_i = a_i \circ b_i + \frac{1}{2}[a_i, b_i]$, where $x \circ y = \frac{1}{2}(xy + yx)$. For an arbitrary element $c \in eRe$ we have $[a_i \circ b_i, c] = [a_i, b_i \circ c] + [b_i, a_i \circ c] \in [eR(1-e), (1-e)Re]$ and $[[a_i, b_i], c] = [a_i, [b_i, c]] - [b_i, [a_i, c]] \in [eR(1-e), (1-e)Re]$. We showed that

$[eRe, eRe] \subseteq [eR(1-e), (1-e)Re]$. The inclusion $[(1-e)R(1-e), (1-e)R(1-e)] \subseteq [eR(1-e), (1-e)Re]$ is proved similarly. Lemma is proved. $\square$

Definition 1. A pair of vector spaces (A^-, A^+) with trilinear products $A^+ \times A^- \times A^+ \to A^+, A^- \times A^+ \times A^- \to A^-$, $a^\sigma \times b^{-\sigma} \times c^\sigma \mapsto (a^\sigma, b^{-\sigma}, c^\sigma) \in A^\sigma$, $\sigma = +$ or $-$, is called an *associative pair* if it satisfies the identities

$$((x^\sigma, y^{-\sigma}, z^\sigma), u^{-\sigma}, v^\sigma) = (x^\sigma, (y^{-\sigma}, z^\sigma, u^{-\sigma}), v^\sigma) = (x^\sigma, y^{-\sigma}, (z^\sigma, u^{-\sigma}, v^\sigma))$$

Example 3. The pair of Peirce components $(eR(1-e), (1-e)Re)$ is an associative pair with respect to the operations $(a^\sigma, b^{-\sigma}, c^\sigma) = a^\sigma b^{-\sigma} c^\sigma$.

Lemma 2. *Let R be a finitely generated algebra and let e, $f \in R$ be idempotents such that $ReR = RfR = R$. Then the associative pair $P = (eRf, fRe)$ is finitely generated.*

Remark 4. In [8] it is proved that if R is a finitely generated algebra, $e \in R$ is an idempotent such that $ReR = R$ then the Peirce component eRe is a finitely generated algebra.

Proof. Suppose that the algebra R is generated by elements $a_1, \dots, a_m$. Suppose further that $a_i = \sum_k \alpha_{ik} u_{ik} e v_{ik} = \sum_t \beta_{it} u'_{it} f v'_{it}$, where $1 \le i \le m$; $\alpha_{ik}, \beta_{it} \in F$; $u_{ik}, v_{ik}, u'_{it}, v'_{it}$ are products in generators $a_1, \dots, a_m$. Let d denote the maximum lengths of the products $u_{ik}, v_{ik}, u'_{it}, v'_{it}$ for all i, k, t. We claim that the pair P is generated by elements euf, fue, where u runs over all products in $a_1, \dots, a_m$ of length $\le 3d+1$. To prove the claim we need to show that for an arbitrary product $u = a_{i_1} \cdots a_{i_N}$ of length $N > 3d+1$ the elements euf, fue lie in the subpair generated by evf, fve, where v runs over all products in $a_1, \dots, a_m$ of length $< N$. There exist integers N_1, N_2, N_3 such that $N/3 - 1 < N_i \le N/3, 1 \le i \le 3$ and

$N = N_1 + N_2 + N_3 + 2$. Let $u = u_1 a_i u_2 a_j u_3$, length $(u_i) = N_i$, $1 \le i \le 3$. Then $a_i = \sum_k \beta_k p'_k f q'_k, a_j = \sum_t \gamma_t p''_t e q''_t$, where $\beta_k, \gamma_t \in F$; p'_k, q'_k, p''_t, q''_t are products in $a_1, \dots, a_m$ of length $\le d$. Now $euf = \sum_{k,t} \beta_k \gamma_t e u_1 p'_k f q'_k u_2 p''_t e q''_t u_3 f$. The lengths of the products $u_1 p'_k, q'_k u_2 p''_t, q''_t u_3$ are less than N. The element fue is treated similarly. Lemma is proved. $\square$

Remark 5. We don't assume that the algebra R of Theorem 1 is unital. However passing to the unital hull we see that if R is a finitely generated algebra, $e \in R$ is an idempotent such that $ReR = R(1-e)R = R$ then the associative pair $(eR(1-e), (1-e)Re)$ is finitely generated.

We will need some definitions from Jordan theory.

Definition 2. An algebra over a field F of characteristic $\neq 2$ with multiplication $a \circ b$ is called a *Jordan algebra* if it satisfies the identities

(J1) $x \circ y = y \circ x$
(J2) $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$.

For references on Jordan algebras see [5, 7, 9].

An arbitrary associative algebra A gives rise to the Jordan algebra

$$A^{(+)} = (A, a \circ b = \frac{1}{2}(ab + ba)).$$

Definition 3. A pair of F-spaces $P = (P^-,\ P^+)$ with trilinear products $P^\sigma \times P^{-\sigma} \times P^\sigma \to P^\sigma$, $a^\sigma \times b^{-\sigma} c^\sigma \to \{a^\sigma, b^{-\sigma}, c^\sigma\} \in P^\sigma$; $\sigma = +$ or $-$; $a^\sigma, c^\sigma \in P^\sigma, b^{-\sigma} \in P^{-\sigma}$, is called a Jordan pair if it satisfies the following identities and all their linearizations:

(J1) $\{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\}$,
(J2) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, z^{-\sigma}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, z^\sigma\}$,
(J3) $\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, z^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}$.

An arbitrary associative pair $A = (A^-, A^+), a^\sigma \times b^{-\sigma} \times c^\sigma \to (a^\sigma, b^{-\sigma}, c^\sigma) \in A^\sigma$ gives rise to the Jordan pair $A^{(+)} = (A^-, A^+)$ with operations $\{a^\sigma, b^{-\sigma}, c^\sigma\} = (a^\sigma, b^{-\sigma}, c^\sigma) + (c^\sigma, b^{-\sigma}, a^\sigma) \in A^\sigma$, $\sigma = +$ or $-$.

For further properties of Jordan pairs see [6].

J.M. Osborn (see [4]) showed that a finitely generated associative algebra R gives rise to the finitely generated Jordan algebra $R^{(+)}$.

Lemma 3. *Let $A = (A^-, A^+)$ be a finitely generated associative pair. Then the Jordan pair $A^{(+)}$ is also finitely generated.*

Proof. Suppose that the pair A is generated by elements $a_1^+, \ldots, a_m^+ \in A^+$; $a_1^-, \ldots, a_m^- \in A^-$. Consider an (associative) product $a = a_{i_1}^+ a_{j_1}^- a_{i_2}^+ \cdots a_{j_s}^- a_{i_{s+1}}^+ \in A^+$. We claim that if the indices $i_1, \ldots, i_{s+1}$ are not all distinct then a is a Jordan expression in shorter products. We have

$$\begin{aligned} x^+y^-u^+v^-u^+ &= (x^+y^-u^+v^-u^+ + u^+v^-x^+y^-u^+) - u^+v^-x^+y^-u^+ \\ &= \{x^+y^-u^+, v^-u^+\} - \frac{1}{2}\{u^+, v^-x^+y^-u^+\} \end{aligned} \tag{1}$$

Linearizing this equality in u^+ (see [9]) we get

$$x^+y^-\{u_1^-, v^-, u_2^+\} = \{x^+y^-u_1^+v^-, u_2^+\} + \{x^+y^-u_2^+, v^-, u_1^+\} - \{u_1^+, v^-x^+y^-, u_2^+\}\,. \tag{2}$$

Now (1) and (2) imply

$$\begin{aligned} x^+y^-u^+v^-u^+z^-t^+ &= x^+y^-(u^+v^-u^+z^-t^+ + t^+z^-u+v^-u^+) - x^+y^-t^+z^-u^+v^-u^+ \\ &= \frac{1}{2}x^+y^-\{\{u^+v^-u^+\},z^-,t^+\} - x^+y^-t^+z^-u^+v^-u^+ \\ &= \frac{1}{2}x^+y^-\{\{u^+v^-u^+\},z^-,t^+\} + \frac{1}{2}\{x^+y^-t^+,z^-,\{u^+,v^-,u^+\}\} \\ &- \frac{1}{2}\{\{u^+,v^-,u^+\},z^-x^+y^-,t^+\} - \{x^+y^-t^+z^-u^+,v^-,u^+\} \\ &+ \frac{1}{2}\{u^+,z^-t^+y^-x^+v^-,u^+\} \end{aligned} \tag{3}$$

If $i_p = i_q$, $p < q$, then a is an expression of the same type as the left hand sides of (1), (3). Hence a is a Jordan expression in shorter products.

We showed that the Jordan pair $A^{(+)}$ is generated by elements $a_{i_1}^+a_{j_1}^-a_{i_2}^+\cdots a_{i_{s+1}}^+$, where the indices $i_1,\dots,i_{s+1}$ are distinct, and elements $a_{j_1}^-a_{i_1}^+\cdots a_{i_s}^+a_{j_{s+1}}$, where $j_i,\dots,j_{s+1}$ are distinct. Hence the pair $A^{(+)}$ is finitely generated. □

Proof of Theorem 1.

Proof. Let R be a finitely generated associative algebra with an idempotent e such that $R = ReR = R(1-e)R$. By Lemma 2 the associative pair $A = (eR(1-e), (1-e)Re)$ is finitely generated. By Lemma 3 the Jordan pair $A^{(+)}$ is finitely generated as well. For arbitrary elements $a^\sigma, c^\sigma \in A^\sigma, b^{-\sigma} \in A^\sigma$, $\sigma = +$ or $-$, we have $\{a^\sigma, b^{-\sigma}, c^\sigma\} = [[a^\sigma, b^{-\sigma}], c^\sigma]$. This implies that generators of the Jordan pair $A^{(+)}$ generate the Lie algebra $A^- + [A^-, A^+] + A^+$. Now it remains to recall that $A^- + [A^-, A^+] + A^+ = [R, R]$ by Lemma 1. Theorem is proved. □

3. Finite generation of Lie algebras $[K, K]$.

Let R, $* : R \to R$, be an involutive algebra with an idempotent e satisfying $ee^* = e^*e = 0$ and $ReR = R(1-e-e^*)R = R$. Let $s = 1-e-e^* \neq 0$. For an arbitrary element $a \in R$ we denote $\{a\} = a - a^* \in K$. Let $R_{-2} = eRe^*$, $R_{-1} = eRs + sRe^*$, $R_0 = eRe + e^*Re^* + sRs$, $R_1 = e^*Rs + sRe$, $R_2 = e^*Re$. Then $R = R_{-2} + R_{-1} + R_0 + R_1 + R_2$ is a $\mathbb{Z}$-grading.

Denote $K_i = K \cap R_i$, $H_i = H \cap R_i$, where $H = \{a \in R | a^* = a\}$.

Remark 6. By Lemma 2 the associative pair (R_{-2}, R_2) is finitely generated. The restriction of $*$ is an involution of the pair (R_{-2}, R_2). However, $-*$ is also an involution of (R_{-2}, R_2). The Jordan pair (K_{-2}, K_2) is $K((R_{-2}, R_2), *) = H((R_{-2}, R_2), -*)$.

An analog of Lemma 3 for associative pairs is not true: the Jordan pair of symmetric elements of a finitely generated involutive associate pair may be not finitely generated. An example can be derived from the Example 2 above.

Lemma 4. *$K_2 = [K_1, K_1]$, $H_2 = span_F\{k^2 | k \in K_1\}$ and, similarly, $K_{-2} = [K_{-1}, K_{-1}]$, $H_{-2} = span_F\{k^2 | k \in K_{-1}\}$.*

Proof. Recall that the algebra R is generated by the elements $a_1, \dots, a_m$. An arbitrary generator a_i can be represented as $a_i = \sum_j \alpha_{ij} v_{ij} s w_{ij}$, where $\alpha_{ij} \in F$; v_{ij}, w_{ij} are products in generators $a_1, \dots, a_m$ (may be, empty). Let $a = a_{i_1} \cdots a_{i_r}$ be an arbitrary product of generators. Applying the equalities above to the generator a_{i_1} we get

$$\begin{aligned} e^* a e &= \sum \alpha_{i_1 j} e^* v_{i_1 j} s w_{i_1 j} a_{i_2} \cdots a_{i_r} e \\ &= \sum \alpha_{i_1 j} \{e^* v_{i_1 j} s\}\{s w_{i_1 j} a_{i_2} \cdots a_{i_r} e\} \in K_1 K_1. \end{aligned}$$

Now $K_2 = \{e^* R e\} = \{K_1 K_1\} = [K_1, K_1]$;

$$\begin{aligned} H_2 &= \{e^* a e + e^* a^* e | a \in R\} \\ &= \{k_1 k_2 + (k_1 k_2)^* | k_1, k_2 \in K_1\} \\ &= \operatorname{span}_F\{k^2 | k \in K_1\}. \end{aligned}$$

Lemma is proved. □

Lemma 5. *There exists a finite subset $M_{-1} \subset K_{-1}$ such that $R_1 = M_{-1} R_2 + R_2 M_{-1}$. Similarly, there exists a finite subset $M_1 \subset K_1$ such that $R_{-1} = M_1 R_{-2} + R_{-2} M_1$.*

Proof. Represent each generator a_i as $a_i = \sum_j \alpha'_{ij} v'_{ij} e w'_{ij}$, where $\alpha'_{ij} \in F$; v'_{ij}, w'_{ij} are products in $a_1, \dots, a_m$(may be, empty). Consider an element $e^* a s$, where $a = a_{i_1} \cdots a_{i_r}$ and apply the decomposition above to a_{i_r}. We'll get

$$\begin{aligned} e^* a s &= \sum \alpha'_{i_r j} e^* a_{i_1} \cdots a_{i_{r-1}} v'_{i_r j} e w'_{i_r j} s \\ &= \sum \alpha'_{i_r j} e^* a_{i_1} \cdots a_{i_{r-1}} v'_{i_r j} \{e w'_{i_r j} s\}. \end{aligned}$$

It remains to choose $M_{-1} = \{\{e w'_{i_r j} s\}\} \subseteq K_{-1}$. Lemma is proved. □

Lemma 6. *(1)* $\sum_{i\neq 0} K_i \subseteq [K,K]$;

(2) $[K,K]$ *is generated by* K_{-1}, K_1.

Proof. (1) For arbitrary elements $k_1 = \{sae\}, k_{-1} = \{sae^*\}$, $a \in R$, we have $k_1 = [k_1, e-e^*]$, $k_{-1} = [e-e^*, k_{-1}]$. Hence $K_{-1}, K_1 \subseteq [K,K]$. By Lemma 4, $K_2 = [K_1, K_1]$, $K_{-2} = [K_{-1}, K_{-1}] \subseteq [K,K]$.

(2) We have $eRe = eRe^*Re = R_{-2}R_2 = K_{-2}K_2 + K_{-2}H_2 + H_{-2}K_2 + H_{-2}H_2$. Hence $\{eRe\} \subseteq [K_{-2}, K_2] + [H_{-2}, H_2] + K_{-2} \circ H_2 + H_{-2} \circ K_2$. This implies that $[\{eRe\}, K_0] \subseteq [K_{-2}, K_2] + [H_{-2}, H_2] \subseteq [K_{-2}, K_2] + [K_{-1}, K_1]$ by Lemma 4. Now sRs is spanned by elements of the type $saebs$; a, $b \in R$. We have $saebs = (sae - e^*a^*s)(ebs - sb^*e^*) - e^*a^*sb^*e^* \in K_1K_{-1} + e^*Re^*$. Hence $\{sRs\} \subseteq [K_{-1}, K_1] + \{eRe\}$. Now $[\{sRs\}, K_0] \subseteq [K_{-1}, K_1] + [K_{-2}, K_2]$ by what we proved above. Lemma is proved.

□

Lemma 7. *Let* $n \geq 2$, $a_2^{(1)}, \dots, a_2^{(n+1)} \in K_2 \cup H_2$; $b_{-2}^{(1)}, \dots, b_{-2}^{(n)} \in K_{-2} \cup H_{-2}$ *and* $Card(\{b_{-2}^{(j)}, j = 1, \dots, n\}) < n$. *Then*

$$a_2^{(1)} b_{-2}^{(1)} \cdots a_2^{(n)} b_{-2}^{(n)} a_2^{(n+1)} \in \sum a_2^{(i_1)} b_{-2}^{(j_1)} a_2^{(i_2)} \cdots b_{-2}^{(j_r)} a_2^{(i_{r+1})} (K_{-2}K_2 + H_{-2}H_2),\ r < n.$$

Proof. Suppose that $a_2^{(n+1)} \in K_2$. If $b_{-2}^{(n)} \in K_{-2}$, then we are done. Let $b_{-2}^{(n)} \in H_{-2}$. Suppose that there exists $1 \leq i \leq n-1$, such that $b_{-2}^{(i)} \in K_{-2}$. Then

$$a_2^{(1)} \cdots b_{-2}^{(i)} \cdots a_2^{(n)} b_{-2}^{(n)} a_2^{(n+1)} = a_2^{(1)} \cdots \{b_{-2}^{(i)} \cdots b_{-2}^{(n)}\} a_2^{(n+1)} \pm a_2^{(1)} \cdots b_{-2}^{(n)} \cdots b_{-2}^{(i)} a_2^{(n+1)}$$

as claimed. Therefore we can assume that $b_{-2}^{(i)} \in H_{-2}$, $1 \leq i \leq n$. Let $a_2^{(n)} \in H_2$. Then

$$b_{-2}^{(n-1)} a_2^{(n)} b_{-2}^{(n)} a_2^{(n+1)} = b_{-2}^{(n-1)} (a_2^{(n)} b_{-2}^{(n)} a_2^{(n+1)} - a_2^{(n+1)} b_{-2}^{(n)} a_2^{(n)}) + b_{-2}^{(n-1)} a_2^{(n+1)} b_{-2}^{(n)} a_2^{(n)}$$

which is an element of $H_{-2}H_2 + H_{-2}K_2H_{-2}H_2$. We will assume therefore that $a_2^{(n)} \in K_2$. Suppose that there exists $2 \leq i \leq n-1$, such that $a_2^{(i)} \in H_2$. Then $a_2^{(i)} \cdots a_2^{(n)} b_{-2}^{(n)} a_2^{(n+1)} = (a_2^{(i)} b_{-2}^{(i)} \cdots a_2^{(n)} + (a_2^{(i)} b_{-2}^{(i)} \cdots a_2^{(n)})^*) b_{-2}^{(n)} a_2^{(n+1)}$ $\pm a_2^{(n)} \cdots a_2^{(i)} b_{-2}^{(n)} a_2^{(n+1)}$. Both summands on the right hand side fall into the case that has just been considered above. From now on we will assume that $a_2^{(2)}, \dots, a_2^{(n+1)} \in K_2$; $b_{-2}^{(1)}, \dots, b_{-2}^{(n)} \in H_{-2}$. Notice that $b_{-2}^{(i-1)} a_2^{(i)} b_{-2}^{(i)} = \{b_{-2}^{(i-1)} a_2^{(i)} b_{-2}^{(i)}\} - b_{-2}^{(i)} a_2^{(i)} b_{-2}^{(i-1)}$. The element $b_{-2} = \{b_{-2}^{(i-1)} a_2^{(i)} b_{-2}^{(i)}\}$ lies in K_{-2}. Hence the product $a_2^{(1)} \cdots a_2^{(i-1)} b_{-2} a_2^{(i+1)} \cdots a_2^{(n+1)}$ is one of those considered before. We proved that the elements $b_{-2}^{(i)}$, $1 \leq i \leq n$, in $a_2^{(1)} b_{-2}^{(1)} \cdots a_2^{(n+1)}$ are skew-symmetric modulo expressions of the desired type. Now taking into account that

$Card(b_{-2}^{(i)},\ 1 \leq i \leq n) < n$, we arrive at the conclusion of the lemma. We started with the assumption that $a_2^{(n+1)} \in K_2$. The case of $a_2^{(n+1)} \in H_2$ is treated similarly. This finishes the proof of the Lemma. $\square$

Proof of Theorem 2.

Proof. By Lemma 2 the associative pair $(R_{-2},\ R_2)$ is finitely generated. Without loss of generality we will assume that (R_{-2}, R_2) is generated by elements $a_2^{(i)}$, $b_{-2}^{(j)} \in K \cup H$, $1 \leq i, j \leq n$. Consider the set of products $P = P_{-2} \cup P_2$,
$P_2 = \{a_2^{(i_1)} b_{-2}^{(j_1)} \cdots a_2^{(i_r)} | 1 \leq r \leq n+2\}$, $P_{-2} = \{b_{-2}^{(j_1)} a_2^{(i_1)} \cdots b_{-2}^{(j_r)} | 1 \leq r \leq n+2\}$.
By Lemma 4 for an arbitrary product $p \in P_{\pm 2}$ we have $p + p^* = \sum \alpha_{p,i} k_{p,i}^2$, where $\alpha_{p,i} \in F$, $k_{p,i} \in K_{\pm 1}$. By Lemma 5 there exist finite sets $M_{-1} \subset K_{-1}$, $M_1 \subset K_1$ such that $R_1 = M_{-1}R_2 + R_2 M_{-1}$, $R_{-1} = M_1 R_{-2} + R_{-2} M_1$.

We claim that the Lie algebra $[K, K]$ is generated by the union of the sets: M_1, M_{-1}, $\{[\{p\}, \{q\}] | p, q \in P\}$, $\{[(p+p^*) \circ k_{q,i},\ k_{q,i}] | p, q \in P\}$, $\{M_{-1}P_2\}, \{M_1 P_{-2}\}$. By Lemma 4 and Lemma 6 it is sufficient to prove that all elements from K_{-1}, K_1 can be expressed by these elements. By Lemma 5 K_1 is spanned by elements of the type $\{k_{-1} a_2^{(i_1)} b_{-2}^{(j_1)} \cdots a_2^{(i_r)}\}$, $k_{-1} \in M_{-1}$. We will use induction on r.

If $r \leq n+2$ then the assumption is clear. If $r > n+2$ then by Lemma 7 applied to $a^{(i_{r-n+2})} b^{(j_{r-n+2})} \ldots a^{(i_{r-1})} b^{(j_{r-1})} a^{(i_r)}$ the element $k_{-1} a_2^{(i_1)} b_{-2}^{(j_1)} \cdots a_2^{(i_r)}$ is a linear combination of elements of the type
$k_{-1} a_2^{(\mu_1)} b_{-2}^{(\nu_1)} \cdots a_2^{(\mu_t)} k'_{-2} k'_2$ and $k_{-1} a_2^{(\mu_1)} b_{-2}^{(\nu_1)} \cdots a_2^{(\mu_t)} h'_{-2} h'_2$, where $t < r$;
$k'_{-2} \in \{P_{-2}\}$, $k'_2 \in \{P_2\}$; h'_{-2}, $h'_2 \in \{p + p^* \mid p \in P\}$. We have

$$\{k_{-1} a_2^{(\mu_1)} b_{-2}^{(\nu_1)} \cdots a^{(\mu_t)} k'_{-2} k'_2\} = [\{k_{-1} a_2^{(\mu_1)} \cdots a_2^{(\mu_t)}\}, [k'_{-2}, k'_2]],$$
$$\{k_{-1} a_2^{(\mu_1)} b_{-2}^{(\nu_1)} \cdots a_2^{(\mu_t)} h'_{-2} h'_2\} = [\{k_{-1} a_2^{(\mu_1)} \cdots a_2^{(\mu_t)}\}, [h'_{-2}, h'_2]]$$

Now it remains to notice that if $h'_2 = p + p^*$, $p \in P$, then $h'_2 = \sum \alpha_{p,\ i} k_{p,\ i}^2$ and $[h'_{-2},\ k_{p,\ i}^2] = 2[h'_{-2} \circ k_{p,\ i},\ k_{p,\ i}] \in [K_{-1},\ K_1]$.

This finishes the proof of the theorem. $\square$

4. Simple algebras

Let R be a simple finitely generated $F-$ algebra with an involution $* : R \to R$, char $F \neq 2$, e is an idemotent such that $ee^* = e^*e = 0$, $K = \{a \in R \mid a^* = -a\}$. If $e + e^*$ is not an identity of R then the Lie algebra $[K, K]$ is finitely generated by theorem 2. Suppose that $e + e^* = 1$. As above, let $R_{-2} = eRe^*$,
$R_0 = eRe + e^*Re^*, R_2 = e^*Re$. Then $R = R_{-2} + R_0 + R_2$ is a $\mathbb{Z}$- grading of R. Denote $K_i = K \bigcap R_i, i = -2, 0, 2$.

If R has a nonzero center Z and $dim_Z R < \infty$ then Z is a finitely generated F-algebra. Since Z is a field it follows that $dim_F Z < \infty$, hence $dim_F R < \infty$ and $dim_F [K, K] < \infty$. From now on we will assume that the algebra R is not finitely dimensional over its center.

Lemma 8. *The algebra R is generated by $K_{-2} + K_2$.*

Proof. Since the algebra R is not finite dimensional over its center it follows that R does not satisfy a polynomial identity. By the result of S. Amitsur [1] R does not satisfy a polynomial identity with involution. Hence $[[[K, K], K], [[K, K], K]] \neq (0)$. I.Herstein [4] proved that if A is a subalgebra of R such that $[A, K] \subseteq A$, A is not commutative and $dim_Z R > 16$, then $A = R$. Applying this result to the associative subalgebra generated by $[[K, K], K]$ we see that $[[K, K], K]$ generates R.

Let us show that $[[K, K], K] \subseteq K_{-2} + [K_{-2}, K_2] + K_2$. The Lie algebra $K_{-2} + [K_{-2}, K_2] + K_2$ is an ideal in the Lie algebra $K = K_{-2} + K_0 + K_2$ and, hence, in the Lie algebra $[K, K]$. As shown by W. E. Baxter [2] the Lie algebra $[K, K]/[K, K] \bigcap Z$ is simple. Hence $[K, K] \subseteq K_{-2} + [K_{-2}, K_2] + K_2 + Z$. Now, $[[K, K], K] \subseteq [K_{-2} + [K_{-2}, K_2] + K_2, K] \subseteq K_{-2} + [K_{-2}, K_2] + K_2$. This finished the proof of the lemma. □

Lemma 9. *Let A be a semi prime $F-$ algebra with in involution $* : A \to A$, $char\ F \neq 2, K = K(A, *) = \{a \in A \mid a^* = -a\}$. Suppose that $k \in K$ and $kKk = (0)$. Then $k = 0$.*

Proof. Let $H = \{a \in A \mid a^* = a\}$. Clearly, $A = K + H$. If $k \neq 0$ then $kAk = kHk \neq (0)$. Choose an element $h \in H$ such that $khk \neq 0$. We have $(khk)K(khk) = (0)$. For an arbitrary element $h_1 \in H$ we have $khkh_1khk = k(hkh_1 + h_1kh)khk - kh_1khkhk = 0$, since $hkh_1 + h_1kh \in K$ and $hkh \in K$. This implies $(khk)A(khk) = (0)$, which contradicts the semi primeness of A. Lemma is proved.

Let $F < X >= F < x_1, ..., x_m >$ be the free associative algebra without 1. The mapping $x_i \to -x_i, 1 \leq i \leq m$, extends to the involution $* : F < X >\to F < X >$. For an arbitrary generator x_i, arbitrary elements $a_1, a_2, a_3 \in K(F < X >, *)$ denote $f(x_i, a_1) = x_i a_1 x_i$, $f(x_i, a_1, a_2) = f(x_i a_1) a_2 f(x_i, a_1)$, $f(x_i, a_1, a_2, a_3) = f(x_i, a_1, a_2) a_3 f(x_i, a_1, a_2)$.

Let I be the ideal of the algebra $F < X >$ generated by all elements $f(x_i, a_1, a_2, a_3), 1 \leq i \leq m, a_1, a_2, a_3 \in K(F < X >), *)$.

Recall that the Baer radical $B(A)$ of an associative algebra A is the smallest ideal of A such that the factor-algebra $A/B(A)$ is semi prime. The Baer radical is locally nilpotent: an arbitrary finite collection of elements from $B(A)$ generates a nilpotent subalgebra (see [3]). □

Lemma 10. *The factor-algebra $F < X > /I$ is nilpotent.*

Proof. Since $I^* = I$, the involution $*$ gives rise to an involution on $A = F < X > /I$. Let $B(A)$ be the Baer radical of A, $\overline{A} = A/B(A)$. The radical $B(A)$ is invariant with respect to any involution, hence $\overline{A}$ is an involutive semi prime algebra. By Lemma 9 the image of an arbitrary element $f(x_i, a_1, a_2, a_3), a_1, a_2, a_3 \in K(F < X >, *)$, is equal to zero in $\overline{A}$. Again, subsequently applying lemma 9 three times we get $f(x_i, a_1, a_2) = 0$ in $\overline{A}$, $f(x_i, a_1) = 0$ in A and, finally, $x_i = 0$ in $\overline{A}$,which means that $A = B(A)$. Since the algebra A is finitely generated we conclude that A is nilpotent. Lemma is proved. □

The degree of nilpotency of the algebra A depends on the number of generators m. Let $F < x_1, ..., x_m >^{d(m)} \subseteq I$.

Now let us return to our finitely generated simple algebra R. By Lemma 8 there exists elements $k_1^+, ..., k_m^+ \in K_2, k_1^-, ..., k_m^- \in K_{-2}$ that generates R.

Lemma 11. *The Jordan pair (K_{-2}, K_2) is generated by elements $\{a\}$, where a are products in $k_i^{\pm}, 1 \leq i \leq m$ generated by products of odd length $< d(2m)$.*

Proof. Choose a generator $k_i^\sigma, \sigma = +\,or\,-$, and three elements $c_1, c_2, c_3 \in K_{-\sigma 2}$. Denote $f_1 = k_i^\sigma c_1 k_i^\sigma, f_2 = f_1 c_2 f_1, f_3 = f_2 c_3 f_2$.
Choose arbitrary elements $a_1, ..., a_n \in K_{-\sigma 2}; b_1, ..., b_n \in K_{\sigma 2}$.

(1) Let $u = a_1 b_1 .. a_n b_n$. Then $f_1 u = k_i^\sigma c_1 k_i^\sigma u = k_i^\sigma c_1 \{k_i^\sigma u\} \pm k_i^\sigma (u^* c_1) k_i^\sigma$. Hence $\{f_1 u\} = \{k_i^\sigma, c_1, \{k_i^\sigma u\}\} \pm \{k_i^\sigma, \{c_1 u\}, k_i^\sigma\}$ is a nontrivial Jordan expression.
(2) Let $u = b_1 a_1 ... b_t a_t, v = a_{t+1} b_{t+1} ... a_n b_n$. Then $\{u f_2 v\} = \{\{u f_1\}, c_2, \{f_1 u\}\} \pm \{f_1 ...\}$ is a nontrivial Jordan expression by (1).
(3) Let $u = a_1 b_1 .. b_{t-1} a_t, v = a_{t+1} b_{t+1} ... a_{n-1} b_{n-1} a_n$, it remains to consider the product $u f_3 v$. We have $f_3 = f_2 c_3 f_2 = f_1 c_2 f_1 c_3 f_1 c_2 f_1 = k_i^\sigma c_1 k_i^\sigma c_2 k_i^\sigma c_1 k_i^\sigma c_3 k_i^\sigma c_1 k_i^\sigma c_2 k_i^\sigma c_1 k_i^\sigma = k_i^\sigma c_2' k_i^\sigma c_3 k_i^\sigma c_2' k_i^\sigma$, where $c_2' = c_1 k_i^\sigma c_2 k_i^\sigma c_1$.

$$\text{Hence, } \{u k_i^\sigma c_2' k_i^\sigma c_3 k_i^\sigma c_2' k_i^\sigma v\} = \{\{u k_i^\sigma c_2'\}, k_i^\sigma c_3 k_i^\sigma, \{c_2' k_i^\sigma v\}\} \pm \{c_2' ...\}$$

The second summand can be treated in the same way as we did in (1).
Now we are ready to finish the proof of the lemma

Let $a = k_{i1}^{\sigma}, k_{j1}^{-\sigma} ... k_{is}^{\sigma}$, $2s-1 \geq d(2m)$. Then by lemma 11 a is a Jordan expression in elements b, where b are products in $k_i^{\pm}, 1 \leq i \leq m$, of odd length less than $2s-1$. This proves the lemma. □

As we have already mentioned above $[K, K]+Z/Z = K_{-2}+[K_{-2}, K_2]+K_2+Z/Z$. In view of Lemma11 this implies that the algebra $[K, K]/[K, K] \bigcap Z$ is finitely generated. Theorem 3 is proved.

Acknowledgement

This paper was funded by King Abdulaziz University, under grant No. (12-130-1434 HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

References

[1] S. A. Amitsur, *Identities in rings with involutions,* Israel 1. Math 7 (1969), 63-68.

[2] W. E. Baxter, Lie simplicity of a special case of associative rings II, Trans. AMS 87(1958), 63-75

[3] I.N Herstein,*Non-commutative Ring*. Carus Mathematical Monographs (revised edition), Vol. 15 Math. Assoc. Amer, Washington, DC (1968)

[4] I.N Herstein, *Rings with Involution*. Mathematics Lecture Notes, University of Chicago,1976

[5] N. Jacobson, Structure and Representations of Jordan Algebras, AMS Coll. Publ., v.39, Providence, 1968

[6] O. Loos, Jordan Pairs, Lecture Notes in Math., v. 460, Springer-Verlag, New York, 1975

[7] K. McCrimmon, A Taste of Jordan Algebras, Springer-Verlag, New York, 2004

[8] S. Montgomery, L. W. Small, Some remarks on affine rings, Proc. AMS 98(1986), 537-544

[9] K. A. Zhevlakov, A. M. Slinko, J. P. Shestakov, A.I. Shirshov, Rings that are Nearly Associative, Academic Press, New York, 1982

a. Department of Mathematics, King Abdulaziz University, Jeddah, SA,
E-mail address, analahmadi@kau.edu.sa; hhaalsalmi@kau.edu.sa:

b. Department of Mathematics, Ohio University, Athens, USA,
E-mail address, jain@ohio.edu:

c. Department of Mathematics, University of California, San Diego, USA,
E-mail address, ezelmano@math.ucsd.edu:

1. To whom correspondence should be addressed, E-mail: ezelmano@math.ucsd.edu.
Author Contributions: A. A., H. A., S. K. J., E. Z. designed research; performed research and wrote the paper. The authors declare no conflict of interest.