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LEAVITT PATH ALGEBRAS OF FINITE GELFAND–KIRILLOV DIMENSION

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Groebner–Shirshov Basis and Gelfand–Kirillov dimension of the Leavitt path algebra are derived.

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1. Introduction

Leavitt path algebras were introduced in [1] as algebraic analogs of graph Cuntz–Krieger C*-algebras. Since then they have received significant attention from algebraists. In this paper we (i) find a Groebner–Shirshov basis of a Leavitt path algebra, (ii) determine necessary and sufficient conditions for polynomially bounded growth, and (iii) find Gelfand–Kirillov dimension.

2. Definitions and Terminologies

A (directed) graph $\Gamma = (V, E, s, r)$ consists of two sets V and E that are respectively called vertices and edges, and two maps s, $r : E \to V$. The vertices $s(e)$ and $r(e)$ are referred to as the source and the range of the edge e, respectively. The graph is called row-finite if for all vertices $v \in V$, $card(s^{-1}(v)) < \infty$ A vertex v for which

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$(s^{-1}(v))$ is empty is called a sink. A path $p = e_1 \dots e_n$ in a graph Γ is a sequence of edges $e_1, \dots, e_n$ such that $r(e_i) = s(e_{i+1})$ for, $i = 1, \dots, n-1$. In this case we say that the path p starts at the vertex $s(e_1)$ and ends at the vertex $r(e_n)$. If $s(e_1) = r(e_n)$, then the path is closed. If $p = e_1 \dots e_n$ is a closed path and the vertices $s(e_1), \dots, s(e_n)$ are distinct, then the subgraph $(s(e_1), \dots, s(e_n); e_1, \dots, e_n)$ of the graph Γ is called a cycle.

Let Γ be a row-finite graph and let F be a field. The Leavitt path F-algebra $L(\Gamma)$ is the F-algebra presented by the set of generators $\{v | v \in V\}, \{e, e^* | e \in E\}$ and the set of relators (1) $v_i v_j = \delta_{v_i, v_j} v_i$ for, all $v_i, v_j \in V$; (2) $s(e)e = er(e) = e, r(e)e^* = e^* s(e) = e^*$ for all $e \in E$; (3) $e^* f = \delta_{e,f} r(e)$, for all $e, f \in E$; (4) $v = \sum_{s(e)=v} ee^*$, for an arbitrary vertex v which is not a sink. The mapping which sends v to v, for $v \in V, e$ to e^* and e^* to e for $e \in E$, extends to an involution of the algebra $L(\Gamma)$. If $p = e_1 \dots e_n$ is a path, then $p^* = e_n^* \dots e_1^*$.

3. A Basis of $L(\Gamma)$

For an arbitrary vertex v which is not a sink, choose an edge $\gamma(v)$ such that $s(\gamma(v)) = v$. We will refer to this edge as special. In other words, we fix a function $\gamma : V \backslash \{\text{sinks}\} \to E$ such that $s(\gamma(v)) = v$ for an arbitrary $v \in V \backslash \{\text{sinks}\}$.

Theorem 1. *The following elements form a basis of the Leavitt path algebra* $L(\Gamma)$: (i) v, *where* $v \in V$, (ii) p, p^*, *where* p *is a path in* Γ, (iii) pq^*, *where* $p = e_1, \dots, e_n$, $q = f_1, \dots, f_m, e_i, f_j \in E$, *are paths that end at the same vertex* $r(e_n) = r(f_m)$, *with the condition that the last edges* e_n *and* f_m *are either distinct or equal, but not special.*

Proof. Recall that a well-ordering on a set is a total order (that is, any two elements can be ordered) such that every non-empty subset of elements has a least element.

As a first step, we will introduce a certain well-ordering on the set of generators $X = V \cup E \cup E^*$. Choose an arbitrary well-ordering on the set of vertices V. If e, f are edges and $s(e) < s(f)$ then $e < f$. It remains to order edges that have the same source. Let v be a vertex which is not a sink. Let $e_1, \dots, e_k$ be all the edges that originate from v. Suppose $e_k = \gamma(v)$. We order the edges as follows: $e_1 < e_2 < \cdots < e_k = \gamma(v)$. Choose an arbitrary well-ordering on the set E^*. For arbitrary elements $v \in V$, $e \in E$, $f^* \in E^*$, we let $v < e < f^*$. Thus the set $X = V \cup E \cup E^*$ is well-ordered. Let X^* be the set of all words in the alphabet X. The length-lex order (see [2, 3]) makes X^* a well-ordered set. For all $v \in V$ and $e \in E$, we extend the set of relators (1)–(4) by (5): $ve = 0$, for $v \neq s(e)$; $ev = 0$, for $v \neq r(e)$; $ve^* = 0$, for $v \neq r(e)$; $e^* v = 0$, for $v \neq s(e), ef = 0$ for $e, f \in E \cup E^*, r(e) \neq s(f)$. The straightforward computations show that the set of relators (1)–(5) is closed with respect to compositions (see [2, 3]). By the Composition–Diamond Lemma ([2, 3]) the set of irreducible words (not containing the leading monomials of relators (1)–(5) as subwords) is a basis of $L(\Gamma)$. This completes the proof. □

4. Leavitt Path Algebras of Polynomial Growth

Recall some general facts on growth of algebras. Let A be an algebra (not necessarily unital), which is generated by a finite dimensional subspace V. Let V^k denote the span of all products $v_1 \cdots v_k, v_i \in V, k \leq n$. Then $V = V^1 \subset V^2 \subset \cdots$, $A = \bigcup_{n\geq 1} V^n$ and $g_{V(n)} = \dim V^n < \infty$. Given the functions f, g from $N = \{1, 2, \ldots\}$ to positive real numbers R_+, we say that $f \preccurlyeq g$ if there exists $c \in N$ such that $f(n) \leq cg(cn)$, for all n. If $f \preccurlyeq g$ and $g \preccurlyeq f$ then the functions f, g are said to be asymptotically equivalent, and we write $f \sim g$. If W is another finite dimensional subspace that generates A, then $g_{V(n)} \sim g_W(n)$. If $g_V(n)$ is polynomially bounded then we define the Gelfand–Kirillov dimension of A as $GKdim\, A = \limsup_{n\to\infty} \frac{\ln g_V(n)}{\ln n}$. The definition of GK-dimension does not depend on a choice of the generating space V as long as $dim\ V < \infty$. If the growth of A is not polynomially bounded, then $GKdim\, A = \infty$.

We now focus on finitely generated algebras and we will assume that the graph Γ is finite. Let C_1, C_2 be distinct cycles such that $V(C_1) \cap V(C_2) \neq \phi$. Then we can renumber the vertices so that $C_1 = (v_1, \ldots, v_m; e_1, \ldots, e_m)$, $C_2 = (w_1, \ldots, w_n; f_1, \ldots, f_n), v_1 = w_1$. Let $p = e_1 \ldots e_m$, $q = f_1 \cdots f_n$.

Lemma 2. *The elements p, q generate a free subalgebra in $L(\Gamma)$.*

Proof. By the Theorem 1 different paths viewed as elements of $L(\Gamma)$ are linearly independent. If u_1, u_2 are different words in two variables, then $u_1(p, q)$ and $u_2(p, q)$ are different paths. Indeed, cutting out a possible common beginning we can assume that u_1, u_2 start with different letters, $u_1(p, q) = p \cdots$, $u_2(p, q) = q \cdots$. If $m > n$ then the path $u_2(p, q)$ returns to the vertex v at the nth step, whereas $u_1(p, q)$ does not. If $m = n$, then the left segments of length m of $u_1(p, q)$, $u_2(p, q)$ are different. This proves the lemma. □

Corollary 3. *If two distinct cycles have a common vertex, then $L(\Gamma)$ has exponential growth.*

From now on we will assume that any two distinct cycles of the graph Γ do not have a common vertex.

For cycles C', C'' we say that $C' \Rightarrow C''$ if there exists a path that starts in $C'$and ends in C''.

Lemma 4. *If C', C'' are two cycles such that $C' \Rightarrow C''$, and $C'' \Rightarrow C'$, then $C' = C''$.*

Proof. Choose a path p that starts in C' and ends in C''. Similarly, choose a path q that starts in C'' and finishes in C'. There exists also a path p' on C'', which connects $r(p)$ with $s(q)$ and a path q' on C', which connects $r(q)$ with $s(p)$. Now, $pp'qq'$ is a closed path, which visits both C' and C''. Let t be a closed path with this property (visiting both C' and C'') having a minimal length. Write $t = e_1 \cdots e_n$, $e_i \in E$. We claim that the vertices $s(e_1), \ldots, s(e_n)$ are all distinct, thus $t = (s(e_1), \ldots, s(e_n); e_1, \ldots, e_n)$ is a cycle. Assuming the contrary, let $s(e_i) = s(e_j)$, $1 \leq i < j \leq n$, and $j - i$ is minimal with this property. Then $t' = (s(e_i), s(e_{i+1}), \ldots, s(e_j); e_i, e_{i+1}, \ldots, e_{j-1})$ is a cycle. Let us "cut it out", that is, consider the path $t'' = e_1 \cdots e_{i-1} e_j \cdots e_n$. This path is shorter than t. Hence t''cannot visit both C' and C''. Suppose that t'' does not visit C'. Then at least one of the vertices $s(e_i), \ldots, s(e_{j-1})$ lies in C'. Since two intersecting cycles coincide, it implies that $t' = C'$, hence $s(e_j)$ lies in C'. This contradicts our assumption that t'' does not visit C'. Hence $t = C' = C''$. This proves the lemma. □

A sequence of distinct cycles $C_1, \ldots, C_k$ is a chain of length k if $C_1 \Rightarrow \cdots \Rightarrow C_k$. The chain is said to have an exit if the cycle C_k has an exit (see [1]), that is, if there exists an edge e such that $s(e) \in V(C_k)$, but e does not belong to C_k.

Let d_1 be the maximal length of a chain of cycles in Γ, and let d_2 be the maximal length of chain of cycles with an exit. Clearly, $d_2 \leq d_1$.

Theorem 5. *Let Γ be a finite graph.*

(1) The Leavitt path algebra $L(\Gamma)$ has polynomially bounded growth if and only if any two distinct cycles of Γ do not have a common vertex;
(2) Under the above assumption $GK\ dim\ L(\Gamma) = \max(2d_1 - 1, 2d_2)$.

Proof. As in the proof of Theorem 1 we consider the generating set $X = V \cup E \cup E^*$ of $L(\Gamma)$. Let E' be the set of edges that do not belong to any cycle. Let P' be the set of all paths that are composed from edges from E'. Then an arbitrary path from P' never arrives to the same vertex twice. Hence, $|P'| < \infty$.

By Theorem 1 the space $Span(X^n)$ is spanned by elements of the following types:

(1) a vertex,
(2) a path $p = p'_1 p_1 p'_2 p_2 \cdots p_k p'_{k+1}$, where p_i is a path on a cycle C_i, $1 \leq i \leq k$, $C_1 \Rightarrow \cdots \Rightarrow C_k$ is a chain, $p'_i \in P'$, $length(p) \leq n$,
(3) p^*, where p is a path of the type (2),
(4) pq^*, where $p = p'_1 p_1 p'_2 p_2 \cdots p_k p'_{k+1}$, $q = q'_1 q_1 q'_2 \cdots q_s q'_{s+1}$; p_i and q_j are paths on cycles C_i, D_j respectively, $C_1 \Rightarrow \cdots \Rightarrow C_k$, $D_1 \Rightarrow \cdots \Rightarrow Ds$ are chains; $p'_i, q'_j \in P', length(p) + length(q) \leq n$ and $r(p) = r(q)$. We will further subdivide this case into two subcases:
   (4.1) $r(p) \notin V(C_k) \cup V(D_s)$;
   (4.2) $r(p) \in V(C_k) \cup V(D_s)$.

We will estimate the number of products of *length* $\leq n$ in each of the above cases and then use the following elementary fact:

Let $(a_n)_n$ be the sum of s sequences $(a_{in})_n$, $1 \leq i \leq s$, $a_{in} > 0$. Then

$$\limsup_{n\to\infty} \frac{\ln a_n}{\ln n} = \max\left(\limsup_{n\to\infty} \frac{\ln a_{in}}{\ln n}, 1 \leq i \leq s\right).$$

Let us estimate the number of paths of the type (2). Fix a chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$. If $C = (v_1, \ldots, v_m; e_1, \ldots, e_m)$ is a cycle, let $P_C = e_1 \cdots e_m$. For a given cycle there are m such paths depending upon the choice of the starting point v_1.

Let $|(V(C_i)| = m_i$ and let P_{C_i} be any one of the m_i paths described above. Then an arbitrary path on C_i can be represented as $u' P_i^l u''$, where $length(u')$, $length(u'') \leq m_i - 1$. Hence every path of the type (2) which corresponds to the chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$ can be represented as $p_1' u_1' P_{C_1}^{l_1} u_1'' \cdots p_k' u_k' P_{Ck}^{l_k} u_k'' p_{k+1}'$, where $p_i' \in P_i'$ and $length(u_i')$, $length(u_i'') \leq m_i - 1$. Clearly, $m_1 l_1 + \cdots + m_k l_k \leq n$. This implies that the number of such paths is asymptotically less than or equal to $n^k \leq n^{d_1}$. On the other hand, choosing a chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_1}$ of length d_1, we can construct $\sim n^{d_1}$ paths of length $\leq n$. The case (3) is similar to the case (2).

Consider now the elements of length $\leq n$ of the type pq^*, $r(p) = r(q)$; the path p passes through the cycles of the chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$ on the way, the path q passes through the cycles of the chain $D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s$ on the way and so $p = p_1' p_1 p_2' \cdots p_k p_{k+1}'$, where $p_i' \in P_i'$, each p_i is a path on the cycle C_i. Similarly, $q = q_1' q_1 q_2' \cdots q_s q_{s+1}'$. Arguing as above, we see that for fixed chains $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$ and $D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s$, the number of such paths is asymptotically less than or equal to n^{k+s}.

Suppose that the vertex $v = r(p) = r(q)$ does not lie in $V(C_k) \cup V(D_s)$. Then both cycles C_k and D_s have exits. Hence the number of paths of type (4.1) is $\leq n^{2d_2}$. On the other hand, let $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_2}$ be a chain and let e be an exit of the cycle C_{d_2}. Select paths $p_2', \ldots, p_{d_2}'$, where p_i' connects C_{i-1} to C_i, $p_i' \in P'$.

Select a path u_1'' on the cycle C_1 which connects $r(P_{C_1})$ to $s(p_2')$, a path u_2' in C_2 which connects $r(p_2')$ to $s(P_{C_2})$, a path u_2'' on C_2 which connects $r(P_{C_2})$ to $s(P_3')$, and so on. The path u_{d_2}'' connects $r(P_{C_{d_2}})$ to $s(e)$.

Among the edges from $s^{-1}(s(e))$, choose a special one $\gamma(s(e))$ different from e. Then by Theorem 1, the elements

$$P_{C_1}^{l_1} u_1'' p_2' u_2' P_{C2}^{l_2} u_2'' p_3' \cdots P_{C_{d_2}}^{l_{d_2}} u_{d_3}'' e e^* (u_2'')^* (P_{C_{d_2}^*})^{l_{d_2+1}} \cdots (u_1'')^* (P_{C_1})^{l_{2d_2}},$$

$$l_i \geq 1, \qquad \text{(A)}$$

are linearly independent. Let m be the total length of all elements other than $P_{C_i}^{l_i}$, $(P_{C_i^*})^{l_{2d_2-i+1}}$. The number of elements described in (A) above is the number of nonnegative integral solutions of the inequality

$$\sum_{i=1}^{d_2} m_i(l_i + l_{2d_2-i+1}) \leq n - m, \quad \text{which is} \sim n^{2d_2}.$$

Now suppose that the vertex $v = r(p) = r(q)$ lies in C_k. Assume at first that $C_k \neq D_s$. Then the chain $D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s$ has an exit. If $k \leq s$, then the number of the paths of this type is $\leq n^{k+s} \leq n^{2d_2}$.

If $s < k$, then $n^{k+s} \leq n^{2k-1} \leq n^{2d_1-1}$.

Next, let $C_k = D_s$. It means that the paths p'_{k+1}, q'_{s+1} are empty; p_k and q_s are both paths on the cycle C_k and in this case we have,

(i) $p_k q_s^* = u$, if $p_k = uq_s$, is a path on C_k,
(ii) $p_k q_s^* = u^*$, if $q_s = up_k$, is a path on C_k, and
(iii) $p_k q_s^* = 0$, otherwise.

The number of such elements pq^* is $\preccurlyeq n^{k+s-1} \leq n^{2d_1-1}$.

On the other hand, let $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_2}$ be a chain of cycles. Select paths $p'_2, \ldots, p'_d \in P', p'_i$ connects C_{i-1} to $C_i; u'_i$, u''_i are paths on the cycle C_i such that $P_{C_1} u''_1 p'_2 u'_2 P_{C_2} u''_2 p'_3 \cdots P_{C_{d_1}} \neq 0$. By Theorem 1, the elements $P_{C_1}^{l_1} u''_1 p'_2 u'_2 P_{C_2}^{l_2} u''_2 p''_3 \cdots u'_{d_1} P_{C_{d_1}}^{l_{d_1}} (u'_{d_1})^* (P^*_{C_{d_1-1}})^{l_{d_1+1}} \cdots (P^*_{C_1})^{l_{2d_1-1}}$ are linearly independent provided that $l_i \geq 1$, $1 \leq i \leq 2d_1 - 1$. The number of these elements is $\sim n^{2d_1-1.}$ This finishes the proof of Theorem 2. □

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