QUASI-PERMUTATION SINGULAR MATRICES ARE PRODUCTS OF IDEMPOTENTS

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Abstract. A matrix $A \in M_n(R)$ with coefficients in any ring R is a quasi-permutation matrix if each row and each column has at most one nonzero element. It is shown that a singular quasi-permutation matrix with coefficients in a domain is a product of idempotent matrices. As an application, we prove that a nonnegative singular matrix having nonnegative von Neumann inverse (also known as generalized inverse) is a product of nonnegative idempotent matrices.

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1. Introduction and preliminaries

Initiated by Erdos (cf. [6]) the problem of decomposing singular matrices into a product of idempotent matrices has been intensively studied by several authors (cf. Fountain [7], Hannah O'Meara [8] and others). Recently it has been shown in [2] that for $n \geq 1$, every $n \times n$ nonnegative singular matrix $A \in M_n(\mathbb{R})$ of rank one has a decomposition into a product of at most three nonnegative idempotent matrices.

A matrix $A \in M_n(R)$ with coefficients in a ring R, is called a quasi-permutation matrix if each row and each column has at most one nonzero element. Using combinatorial techniques, we show that singular quasi-permutation matrices with coefficients in any domain can always be represented as a product of idempotent matrices (Theorem 7).

As an application, we show that nonnegative matrices having nonnegative von Neumann inverse (also known as generalized inverse) can be decomposed into a product of nonnegative idempotents (Theorem 16). Indeed, the well-known structure of nonnegative idempotent matrices and the structure of nonnegative matrices that have a nonnegative von Neumann inverse reveal strong links with rank one matrices and the quasi-permutation matrices (cf. [11]). We make use of their structure to establish our results.

For convenience, we state below two lemmas that are used often in the proofs of our results [2].

Lemma 1. *Any singular nonnegative matrix of rank 1 can be presented as a product of three nonnegative idempotent matrices of rank one.*

Lemma 2. *Any nonnegative nilpotent matrix is a product of nonnegative idempotent matrices.*

2. Quasi-permutation matrices

Let us recall that for a permutation $\sigma \in S_n$, the permutation matrix P_σ associated with σ is an $n \times n$ matrix defined by

$$P_\sigma = \sum_{i=1}^{n} e_{i,\sigma(i)}.$$

We consider a more general situation as given in the following definition.

Definition 3. *A matrix $A \in M_n(R)$ with coefficients in a ring R will be called a quasi-permutation matrix if each row and each column has at most one nonzero element.*

Remarks 4.
(a) A quasi-permutation matrix can be singular and, in this case, it has at least one zero row and one zero column. We will mainly work with rows but the analogous properties for columns also hold (acting on the right with given permutation matrices).
(b) Particularly important for our purposes will be the quasi-permutation matrices, denoted by $P_{\sigma,l}$, $\sigma \in S_n$, $l \in \{1, \ldots, n\}$*obtained from* P_σ by changing the nonzero element of the l^{th} row of P_σ to 0. We thus have

$$P_{\sigma,l} = \sum_{\substack{i=1\\ i\neq l}}^{n} e_{i,\sigma(i)}.$$

(c) We observe that the $\sigma(l)^{th}$ column of the matrix $P_{\sigma,l}$ is the only column that is zero.

We give some properties of the matrices $P_{\sigma,l}$ in the following lemma.

Lemma 5. *Let σ, $\tau \in S_n$ and $l \in \{1, \ldots, n\}$. Then*
a) $P_\sigma P_{\tau,l} = P_{\tau\sigma,\sigma^{-1}(l)}$
b) $P_{\tau,l} \cdot P_\sigma = P_{\sigma\tau,l}$
c) $P_\sigma P_{\tau,l} P_\sigma^{-1} = P_{\sigma^{-1}\tau\sigma,\sigma^{-1}(l)}$
d) *If $\sigma \in S_n$ has no fixed point, then for any $1 \leq l \leq n$, $P_{\sigma,l}$ is a nilpotent matrix. In particular, if $c = (1, \ldots, n)$ is the cycle of length n defined in the usual way, then $P_{c,l}$ is a nilpotent matrix. Moreover, when considered as real matrices, $P_{\sigma,l}$ and $P_{c,l}$ are product of nonnegative idempotent matrices (i.e. the coefficients of the idempotent matrices are in $M_n(\mathbb{R}^+)$).*

Proof. We will only prove statement d). Let $\sigma, \tau \in S_n$ and $1 \leq l, s \leq n$. We can easily compute that $P_{\sigma,l}P_{\tau,s} = \sum_{i\neq l,\sigma(i)\neq s} e_{i,\tau(\sigma(i))}$. In case $\tau = \sigma$ has no fixed point and $l = s$, then $P_{\sigma,l}^2$ has two zero rows. Similarly, we get that $P_{\sigma,l}^3$ has three zero rows and continuing this process we easily conclude that $P_{\sigma,l}$ is nilpotent (of index bounded by n)

The last statement of d) above comes from the fact that nilpotent nonnegative matrices are always product of nonnegative idempotent matrices. □

The following somewhat technical proposition will be very useful while proving the main result of this section.

Proposition 6. *Let R be any ring, $\sigma \in S_n$ and $A \in M_n(R)$ be a quasi-permutation matrix having its l^{th} row and its r^{th} column equal to zero. Then*

a) *The $\sigma^{-1}(l)^{th}$ row of $P_\sigma A$ is zero and we have $P_\sigma A = P_{\sigma,\sigma^{-1}(l)}A$. Similarly, the $\sigma(r)^{th}$ column of AP_σ is zero and we have $AP_\sigma = AP_{\sigma,r}$.*
b) *$A = P_{\sigma^{-1},l}(P_\sigma A)$ and $A = (AP_\sigma)P_{\sigma^{-1},\sigma(r)}$.*
c) *Suppose that $P_{\sigma^{-1},l}$ (resp. $P_{\sigma^{-1},\sigma(r)}$) is a product of idempotent matrices. If the same is true for $P_\sigma A$ (resp. for AP_σ) then A itself is a product of idempotent matrices.*

Proof. a) The proof is easy.
b) We have $A = P_{\sigma^{-1}}(P_\sigma A) = P_{\sigma^{-1},l}(P_\sigma A)$ by the above statement a). The other equality follows similarly.
c) This is clear from the statement b) above.

□

We are now ready to state and prove the main result of this section.

Theorem 7. (a) *Let R be any domain and $A \in M_n(R)$ be a singular quasi-permutation matrix. Then A is a product of idempotent matrices.*
(b) *Any singular nonnegative quasi-permutation matrix $A \in M_n(\mathbb{R}^+)$ is a product of nonnegative idempotent matrices.*

Proof. We prove both statements by induction on $n \geq 1$. (a) Since A is a singular quasi-permutation matrix, $A = 0$ if $n = 1$. Let $n > 1$ and assume that the result holds for any singular quasi-permutation matrix of size smaller than n. Let $c = (1, \ldots, n)$ be a cyclic permutation. It has been shown in Lemma 5 that, considered over the reals, $P_{c,l}$ is a product of nonnegative idempotent matrices. Let us now show that, for any $1 \leq l \leq n$, the quasi-permutation matrix $P_{c,l}$ is a product of idempotent matrices. If $l \neq n$, we let $\sigma = (l, n) \in S_n$ be the transposition exchanging l and n, and if $l = n$ we let σ be the identity. The property (c) in Lemma 5 implies that $P_\sigma P_{c,l} P_{\sigma^{-1}} = P_{\sigma^{-1}c\sigma,n}$ has its last row zero. Clearly, if $l = n$, $P_{c,n}$ has its last column nonzero and, if $l \neq n$, the remark 4 (c) implies that the last column of $P_{\sigma^{-1}c\sigma,n}$ is also not zero, so that, in any case, $P_\sigma P_{c,l} P_{\sigma^{-1}}$ is a quasi-permutation matrix of the form

$$P_\sigma P_{c,l} P_{\sigma^{-1}} = \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix}$$

where B is a quasi-permutation matrix. Since a column of B is zero, B is singular and our induction hypothesis shows that B is a product of idempotent matrices, say $B = E_1 \cdots E_s$ where, for $i = 1, \ldots, s$, $E_i^2 = E_i \in M_n(R)$. This implies that

$$P_\sigma P_{c,l} P_\sigma^{-1} = \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{n-1} & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{n-1} & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} E_s & 0 \\ 0 & 1 \end{bmatrix}$$

and hence, $P_{c,l}$ is also a product of idempotent matrices. Let us now come back to a general singular quasi-permutation matrix $A \in M_n(R)$. There exists $1 \leq l \leq n$ such that the l^{th} row of A is zero. Let c be the cycle $(1, \ldots, n)$. There exists $r \in \{0, 1, \ldots, n-1\}$ such that $P_{c^r}A$ has its bottom row $(0, \ldots, 0)$.

Similarly, acting on the columns, we may assume that there exists $0 \leq s \leq n-1$ such that $P_{c^r}AP_{c^s}$ has a zero column which is *not* the last one. Let us notice that after permuting cyclically the rows and the columns of a quasi-permutation matrix, we still obtain a permutation matrix. In other words, we may assume that $P_{c^r}AP_{c^s}$ is of the form

$$P_{c^r}AP_{c^s} = \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} \quad , \quad B \in M_{n-1}(\mathbb{R}^+)$$

where B is a quasi-permutation matrix. Since a column of B is zero, B is singular and our induction hypothesis shows that B is a product of idempotent matrices. As in the previous paragraph , we conclude that $P_{c^r}AP_{c^s}$ is a product of idempotent matrices. Since the l^{th} row of A is zero, the same is true for the l^{th} row of AP_{c^s}. On the other hand, the first paragraph of this proof shows that $P_{c^r,l}$ is a product of idempotent matrices hence, applying proposition 6 c), we obtain that AP_{c^s} is a product of idempotent matrices. The singular quasi-permutation matrix A also has a zero column and the same is true for AP_{c^s}. Repeating the same arguments and using 6 c) again, we easily conclude that A is a product of idempotents, as desired.

(b) The proof of the part b) follows exactly the same pattern as the proof of part a), using the fact that $P_{c,l}$ is a product of nonnegative idempotent matrices (cf. Lemma 5 (d)). □

Remark 8. The hypothesis that the ring R is a domain in part (a) of the above theorem is only necessary to start the induction process. If the ring R is assumed to be such that the left and right zero divisors are product of idempotent elements, then part (a) of the theorem holds in this case.

The following definition somewhat enlarges the definition of a quasi-permutation matrix.

Definition 9. *Let R be a ring. A matrix $A \in M_n(R)$ is a quasi-permutation block matrix if there exist a sequence of natural numbers $n_1, \dots, n_l$ such that $n_1 + \cdots + n_l = n$ and a permutation $\sigma \in S_l$ such that $A = (A_{ij})$ where A_{ij} are matrices of size $n_i \times n_{\sigma^{-1}(j)}$ and $A_{ij} = 0$ if $j \neq \sigma(i)$.*

Remarks 10. (1) As a consequence of the definition, the nonzero blocks $A_{i\sigma(i)}$ of a quasi-permutation block matrix are square matrices of size $n_i \times n_i$.
(2) Notice also that a quasi-permutation block matrix whose all nonzero entries are quasi-permutation matrices must itself be a quasi-permutation matrix.

Next we prove a generalization of the theorem 7.

Proposition 11. *Let $A = (A_{ij})$, $1 \leq i, j \leq l$ be a quasi-permutation block matrix associated with the permutation $\sigma \in S_l$. Suppose that, for every i, $1 \leq i \leq l$, there exist matrices $E_i, B_i, F_i \in M_{n_i \times n_i}(\mathbb{R}^+)$ such that $A_{i\sigma(i)} = E_i B_i F_i$. Then A can be factorized in the following way:*

$$A = diag(E_1, \dots, E_l)(B)diag(F_{\tau(1)}, \dots, F_{\tau(l)}) \qquad (*)$$

where $\tau = \sigma^{-1}$ and the matrix $B = (B_{ij})$ is a quasi-permutation block matrix associated with σ such that $B_{i\sigma(i)} = B_i$.

Moreover, if for $1 \leq i \leq n$ the blocks B_i are quasi-permutation matrices, then A is a product of idempotent matrices.

Proof. Let D be the matrix on the right hand side of the equality (*). Let us compute the (s, r) block of D, for $1 \leq r, s \leq l$.

We have

$$\begin{aligned}
D_{sr} &= \big(diag(E_1,\ldots,E_l)(B_{ij})diag(F_{\tau(1)},\ldots,F_{\tau(l)})\big)_{s,r}\\
&= \sum_{k=1}^{l}(diag(E_1,\ldots,E_l)_{sk}\big((B_{ij})diag(F_{\tau(1)},\ldots,F_{\tau(l)})\big)_{kr}\\
&= E_s\big((B_{ij})diag(F_{\tau(1)},\ldots,F_{\tau(l)})\big)_{sr}\\
&= E_s\big(\sum_t (B_{ij})_{st}diag(F_{\tau(1)},\ldots,F_{\tau(l)})_{tr}\big)\\
&= E_sB_{s\sigma(s)}(diag(F_{\tau(1)},\ldots,F_{\tau(l)})_{\sigma(s)r}
\end{aligned}$$

If $r \neq \sigma(s)$ we get $(diag(F_{\tau(1)1},\ldots,F_{\tau(l)l})_{\sigma(s)r} = 0$ and hence $D_{sr} = 0$. If $r = \sigma(s)$ we have $D_{sr} = E_sB_sdiag(F_{\tau(1)},\ldots,F_{\tau(l)})_{\sigma(s)\sigma(s)} = E_sB_sF_{\tau(\sigma(s))} = E_sB_sF_s = A_{s\sigma(s)} = A_{sr}$. This shows that $D_{sr} = A_{sr}$ for all $1 \leq r, s \leq l$.

For the additional statement we first remark that (as mentioned in the remark 10 (2) above) a quasi-permutation block matrix whose blocks are quasi-permutation matrices is itself a quasi-permutation matrix. The result then follows immediately from Theorem 7. □

3. Application

Our aim is to use the results related to the quasi-permutation matrices obtained in the previous section to prove that a nonnegative singular matrix with nonnegative von-Neumann inverse is a product of nonnegative idempotent matrices. All the matrices in this section will be real matrices.

Let us first state a result which will be used in the application.

Proposition 12. *(cf. [11], Theorem 1 and Lemma 2) If a nonnegative square matrix A admits a nonnegative von Neumann inverse X (ie. $A = AXA$) then there exists a permutation matrix P such that PAP^T is of the form*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0\\ 0 & 0 & 0 & 0\\ CJ & CJD & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where C and D are nonnegative matrices of suitable sizes and J is a direct sum of matrices of the following two types.

1) βxy^T where x, y are positive vectors and β is a postive real number.

2)

$$\begin{bmatrix} 0 & \beta_1x_1y_1^T & 0 & 0 & \cdots & 0\\ 0 & & \beta_2x_2y_2^T & 0 & \cdots & 0\\ \vdots & & & \ddots & \cdots & \vdots\\ 0 & & & 0 & \cdots & \beta_{d-1}x_{d-1}y_{d-1}^T\\ \beta_dx_dy_d^T & & & 0 & \cdots & 0 \end{bmatrix}$$

where for $1 \leq i \leq d$ the vectors x_i, y_i are positive and β_i is a positive real number.

The following lemma is straightforward.

Lemma 13. *Let A be a nonnegative matrix having a nonnegative quasi-inverse and let J, C, D be matrices associated with A as in the above proposition 12. If J is a product of nonnegative idempotent marices then the same is true for A.*

This lemma shows that, in order to prove that singular nonnegative matrices having nonnegative quasi-inverse are product of nonnegative idempotents, it is enough to show that singular matrices of type J as described in the above proposition 12 are product of nonnegative idempotent matrices.

Remark 14. We have seen that a nonnegative matrix A with nonnegative von Neumann inverse has a very special form (cf. Proposition 12). The two types of matrices appearing in the description of A are clearly quasi-permutation block matrices. Since these different types are all located on the diagonal of A, the matrix A itself is a quasi-permutation block matrix. Moreover, in this case, the nonzero blocks are nonnegative matrices of rank one.

Lemma 15. *Let $E^2 = E \in M_n(\mathbb{R}^+)$, $n \geq 2$, be an idempotent matrix of rank 1. Then there exists a matrix N such that $E = ENE$ where N is a singular quasi-permutation matrix with nonnegative entries.*

Proof. Since the matrix $E = E^2$ is of rank one, there exist vectors $a = (a_1, \dots, a_n)^T$, $b = (b_1, \dots, b_n)^T \in \mathbb{R}^n$ such that $E = ab^T$ and $b^T a = 1$. In particular, there exists $1 \leq l \leq n$ such that $a_l b_l > 0$. Let $N = (N_{ij}) \in M_n(\mathbb{R})$ be the matrix defined by $N_{ij} = (a_l b_l)^{-1}\delta_{il}\delta_{jl}$ where δ is the classical Kronecker symbol. It is easy to check that $b^T N a = 1$ and hence $ENE = ab^T Nab^T = ab^T = E$. Obviously the matrix N is a quasi-permutation matrix. □

We are now ready to prove the main result of this section.

Theorem 16. *Let A be a singular nonnegative matrix having a nonnegative von Neumann inverse X (i.e. $A = AXA$). Then A is a product of nonnegative idempotent matrices.*

Proof. According to Remark 14 A is a quasi-permutation block matrix associated with a permutation $\sigma \in S_l$. We can thus write $A = (A_{ij})$ where the nonzero blocks $A_{i\sigma(i)}$ are matrices of rank 1. Hence for any $1 \leq i \leq l$ we have either $A_{i\sigma(i)}$ is the zero matrix or a scalar or it is a singular square matrix of rank one (and size bigger than 2). In any case, using Lemma 1, we can write $A_{i\sigma(i)} = E_i B_i F_i$ where E_i and F_i are idempotent matrices and B_i is either a scalar or an idempotent matrix of rank 1. Proposition 11 shows that our matrix A can be written as $A = diag(E_1, \dots, E_l)Bdiag(F_{\tau(1)}, \dots, F_{\tau(l)})$, where $\tau = \sigma^{-1}$. The two diagonal matrices on the right hand side of this equality are idempotent matrices and the matrix B is a quasi-permutation block matrix associated with σ where the nonzero blocks $B_i = B_{i\sigma(i)}$ are either scalar or nonnegative idempotent matrices of rank one. Hence, for any $1 \leq i \leq l$, using Lemma 15, we can write $B_i = E_i N_i E_i$ where E_i is a nonnegative idempotent matrix ($E_i = 1$ if B_i is a scalar) and N_i is either a scalar (if B_i is a scalar, $B_i = N_i$) or a singular quasi-permutation matrix with nonnegative entries. We thus conclude that, for any $1 \leq i \leq l$, the matrix N_i is a quasi-permutation matrix. Proposition 11 implies that $B = diag(E_1, \dots, E_l)Ndiag(E_{\tau(1)}, \dots, E_{\tau(l)})$, where

$\tau = \sigma^{-1}$. Finally, we can write $B = eNf$ where e and f are idempotent matrices and the matrix N is a quasi-permutation block matrix whose nonzero blocks are quasi-permutation matrices, and so N itself is a quasi-permutation matrix (cf. Remark 10 (2)).
Notice that since A is singular there must exist at least one integer $r \in \{1, \ldots, l\}$ such that $A_{r,\sigma(r)}$ is singular. Hence there must exist r such $A_{r,\sigma(r)}$ is either zero or a singular matrix of rank 1. This implies that there exists $1 \leq r \leq l$ such that B_r is either zero or a singular idempotent matrix of rank 1. We have seen above that the matrix B can be decomposed as $B = eNf$ where e and f are idempotent matrices and N is a singular quasi-permutation matrix. Theorem 7 implies that N is a product of nonnegative idempotent matrices and hence B and finally A is also a product of nonnegative idempotent matrices, as required. □

Remark 17. Let us remark that the positivity of the vectors x_i, y_i that appears in the second type of the description of the matrices J (cf. Proposition 12) is not used in the proof. We only need that these vectors are nonnegative.

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