# MATH3200: APPLIED LINEAR ALGEBRA
# PRACTICE MODULE 24: TOOLS FOR SOLVING LINEAR SYSTEMS: MATRICES IN ROW ECHELON FORM AND BACK-SUBSTITUTION

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We will use here the notation and terminology of Lecture 13.

## 1. Row-reduced matrices

In Conversation 12 and at the beginning of Lecture 13 we considered the following extended matrices of systems of linear equations:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Let us take a closer look at their common features. The second of these matrices contains a *zero row,* that is, a row that consists entirely of zeros. Note that this zero row appears as the last row of that matrix. The first entry in each *nonzero row* of each of these matrices is a 1, with only zeros appearing in the same column below that 1. These ones appear further and further to the right in subsequent rows. Matrices of this form are said to be in *row echelon form* or simply *echelon form.* Recall the formal definition of this property from Lecture 13:

**Definition 1.** *A matrix is in* row echelon form *or simply* echelon form *if:*

- (R1) *All* zero rows, *that is, rows with only zeros, appear below all* nonzero rows *when both types are present.*
- (R2) *The first nonzero entry in any nonzero row is 1.*
- (R3) *All elements in the same column below the first nonzero element of a nonzero row are 0.*
- (R4) *The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.*

Systems with extended matrices in row echelon form fairly easy to solve, either by directly reading off the solution, or by back-substitution.

In Lecture 13, we also saw a stronger version of the row echelon form. It is defined as follows:

**Definition 2.** *A matrix is in* reduced row echelon form *or simply* reduced echelon form, *if:*

- (R1) *All* zero rows, *that is, rows with only zeros, appear below all* nonzero rows *when both types are present.*
- (R2) *The first nonzero entry in any nonzero row is 1.*
- (R3+) *All elements in the same column* as *the first nonzero element of a nonzero row are 0.*
- (R4) *The first nonzero element in a nonzero row appears in a column further to the right of the first nonzero element in any preceding row.*

Note that the only difference between the definitions of echelon form and reduced echelon form of a matrix is that we replaced the word "below" in the definition of condition (R3) with the word "as" in condition (R3+).

**Question 24.1:** Consider the following matrices. Which of them are in row echelon form?

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 & -4 & 3 \\ 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 & -4 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 2 & -4 & 3 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

**Question 24.2:** Consider the five matrices in row echelon form that we already saw at the beginning of this module:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1 & -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Which of these matrices is/are in reduced row-echelon form?

## 2. Back-substitution

Linear system whose extended matrices are in row echelon form can be solved by *back-substitution.* The idea is to read the value of the last variable right off the last equation, substitute the value back into the second-last equation and then solve for the second last variable, substitute these two values into the third last equation, and so on, until you can solve the first equation for the first variable.

For example, $[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ is the extended matrix of the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & 4x_3 & = & 0 \\ & & & & x_3 & = & 5 \end{array}$$

By substituting $x_3 = 5$ into the second equation we get $x_2 + 20 = 0$, so that $x_2 = -20$, and when we substitute these two values for $x_2$ and $x_3$ into the first equation we get $x_1 - 40 + 15 = 4$, so that $x - 1 = 29$.

In other words, the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 29 \\ -20 \\ 5 \end{bmatrix}$ is the unique solution of this system.

When doing back-substitution, it is best to translate extended matrices first back into the systems that they represent to avoid any confusion. For example, the extended matrices

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad [\mathbf{B}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

look very similar, but one of them represents an inconsistent system, while the other one represents a system with a unique solution. You can see this by translating them back into the systems that they represent:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & 5x_3 & = & 6 \\ & & & & x_3 & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & 5x_3 & = & 6 \\ & & & & 0 & = & 1 \end{array}$$

**Question 24.3:** Which of the above matrices $[\mathbf{A}, \vec{\mathbf{b}}]$ and $[\mathbf{B}, \vec{\mathbf{b}}]$ represents an inconsistent system, and what is the solution of the other system?

Things become a little more complicated if there are zero rows in the extended matrix, or, more generally, if there are fewer nonzero rows than there are variables. Consider the following example:

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix represents the following system with variables $x_1, x_2, x_3, x_4$:

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 4 \\ & & & & x_3 & - & x_4 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

The last line doesn't give us any information about $x_4$, and all we can do is work with $x_4$ in symbolic form, by leaving it as a so-called *free variable.* From the second equation we then get $x_3 = x_4$, which we can substitute in the first equation. This gives: $x_1 + 2x_2 + 3x_4 + 4x_4 = 4$, or $x_1 = 4 - 2x_2 - 7x_4$. This tells us that we can choose *any* numbers for the two free variables $x_2$ and $x_4$, and these two chosen numbers will then uniquely determine the values of $x_1$ and $x_3$. This system is *underdetermined;* it has infinitely may solutions. The solution set consists of all vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 2x_2 - 7x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix}$$

There is more than one way to write the solution set here. For example, we could have chosen $x_2$ and $x_3$ as our free variables instead and written the solution set as the set of all vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 2x_2 - 7x_3 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix}$$

**Question 24.4:** Find the solution set of the system that is represented by the extended matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Question 24.5:** Find the solution set of the system that is represented by the extended matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

**Question 24.6:** Find the solution set of the system that is represented by the extended matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

**Question 24.7:** Find the solution set of the system that is represented by the extended matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Question 24.8:** Find the solution set of the system that is represented by the extended matrix

$$[\mathbf{A}, \vec{\mathbf{b}}] = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$