# MATH3200: APPLIED LINEAR ALGEBRA
# SELF-STUDY AND PRACTICE MODULE 71: APPLICATIONS OF LEFT EIGENVECTORS TO MARKOV CHAINS

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This module is based on Lectures 36, 38, Conversations 35, 36, and Modules 67B, 68.

The module also uses MatLab. Start a MatLab session now.

For easier reference, let us again briefly review some basic facts about Markov chains.

A Markov chain is a *stochastic process* where time proceeds in discrete steps $t = 0, 1, 2, \dots$.
At each time $t$ the process can be in exactly one of several *states* that are numbered $1, \dots, n$.
The probability of being in a given state at time $t+1$ depends *only* on the state at time $t$.
A matrix $\mathbf{P} = [p_{ij}]_{n\times n}$ gives the *transition probabilities* $p_{ij}$ from state $i$ at time $t$ to state $j$ at time $t+1$.
When $\vec{\mathbf{x}}(t) = [x_1(t), \dots, x_n(t)]$ is the *probability distribution* for the states at time $t$, then the probability distribution $\vec{\mathbf{x}}(t+1)$ at time $t+1$ is given by

$$\vec{\mathbf{x}}(t+1) = \vec{\mathbf{x}}(t)\mathbf{P} = [x_1(t), \dots, x_n(t)]\mathbf{P}. \tag{1}$$

The matrix $\mathbf{P}$ is a *stochastic matrix,* which means that each of its rows adds up to 1. There always exists a left eigenvector $\vec{\mathbf{x}}$ with eigenvalue $\lambda = 1$ of $\mathbf{P}$ that is a probability distribution. Such a vector is called a *stationary distribution* of the Markov chain. It has the property that if $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}$ at some time $t$, then $\vec{\mathbf{x}}(t+1) = \vec{\mathbf{x}}(t)$.

In our example weather.com light we assumed that state 1 means "sunny day" and state 2 means "rainy day." The matrix of transition probabilities then takes the form $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ where:

- $p_{11}$ is the probability that a sunny day is followed by another sunny day.
- $p_{12}$ is the probability that a sunny day is followed by a rainy day.
- $p_{21}$ is the probability that a rainy day is followed by a sunny day.
- $p_{22}$ is the probability that a rainy day is followed by another rainy day.

Probability distributions at time $t$ are then row vectors $\vec{\mathbf{x}}(t) = [x_1(t), x_2(t)]$, where

- $x_1(t)$ is the probability that day $t$ will be a sunny day.
- $x_2(t)$ is the probability that day $t$ will be a rainy day.

Now consider weather.com light with $\mathbf{P} = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{bmatrix}$

We found in Conversation 36 that the probability distribution $\vec{\mathbf{x}} = [1/3, 2/3]$ is a left eigenvector with eigenvalue 1 of $\mathbf{P}$. We can verify this with MatLab:

```
>> P = [0.4, 0.6; 0.3, 0.7]
>> [1/3, 2/3]*P
```

The output will be the same vector $\vec{\mathbf{x}}$, shown with an accuracy of four decimal places.

For the remainder of this module, it will be useful to see more decimal places of MatLab.
So let's switch to a different format and then look at the output of the last command again:

```
>> format long
>> ans
```

As we learned in Module 68, we can find the transposes of a full set of left eigenvectors of $\mathbf{P}$ by entering:

```
>> [vec, val] = eig(P')
```

This will show us an eigenvector of $\mathbf{P}^T$ with eigenvalue 1 that is scaled version of the transpose of the vector $\vec{\mathbf{x}} = [1/3, 2/3]$, that is, of the stationary probability distribution that we just explored. Moreover, it will show us that $\mathbf{P}^T$ also an eigenvector with eigenvalue $\lambda_2 = 0.1$. The transpose of such an eigenvector will be a left eigenvector of $\mathbf{P}$. After rescaling it we find thu $\vec{\mathbf{y}} = [-1, 1]$ will be a left eigenvector of $\mathbf{P}$ with eigenvalue 0.1. But since its coordinates are of opposite signs, we cannot rescale $\vec{\mathbf{y}}$ to a probability distribution. It plays a quite interesting role in the behavior of a Markov chain, but exploring this role goes beyond the scope of this course.
One can deduce from the above information about the left eigenvectors of $\mathbf{P}$ that $\vec{\mathbf{x}} = [1/3, 2/3]$ is the only stationary probability distribution of the corresponding Markov Chain. Let us explore numerically in MatLab what happens if we start with a vector $\vec{\mathbf{x}}(0) \neq [1/3, 2/3]$. Let's start with a rainy day $\vec{\mathbf{x}}(0) \neq [0, 1]$:

```
>> [0,1]*P
>> ans*P
>> ans*P
```

Repeat a few more times to watch what happens, and then do the same for a sunny day as your starting point:

```
>> [1,0]*P
>> ans*P
>> ans*P
```

And so on. What you will see is sequences of the probability distributions for consecutive days,

$$[x_1(0), x_2(0)], [x_1(1), x_2(1)], [x_1(2), x_2(2)], \dots, [x_1(t), x_2(t)], \dots$$

These vectors get closer and closer to the stationary distribution $\vec{\mathbf{x}} = [1/3, 2/3]$ and eventually you will no longer see any change in their values at the accuracy that is shown in the output.

**Question 71.1:** How many time steps does it take until you see a result that does no longer change on the screen? Be sure to use `>> format long` for this question.

These observations illustrate the following general theorem:

**Theorem 1.** *Let $\mathbf{P}$ be the transition probability matrix of a Markov chain with $n$ states and let $\vec{\mathbf{x}} = [x_1, x_2, \dots x_n]$ be a stationary distribution for this Markov chain. Assume that the eigenvalue $\lambda^* = 1$ of $\mathbf{P}$ has multiplicity 1 and all other eigenvalues $\lambda$ of $\mathbf{P}$ have absolute value $|\lambda| < 1$. Then if $\vec{\mathbf{x}}(0)$ is any initial distribution, the distributions $\vec{\mathbf{x}}(t)$ always approach $\vec{\mathbf{x}}$ as $t \to \infty$.*

The assumption that the eigenvalue $\lambda^* = 1$ of $\mathbf{P}$ has multiplicity 1 and all other eigenvalues $\lambda$ of $\mathbf{P}$ have absolute value $|\lambda| < 1$ of Theorem 1 may look a little puzzling. Let us see why it is needed. Let us explore a Markov chain for weather.com light with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Question 71.2:** Use MatLab to find:

(a) All eigenvalues of $\mathbf{P}$.
(b) All stationary distributions of the Markov chain that is defined by $\mathbf{P}$.

In answering Question 71.2 you will find that the assumptions of Theorem 1 are not satisfied for this Markov chain. Now explore what happens here when we start with a sunny day:

```
>> P = [0, 1; 1, 0]
>> [1,0]*P
>> ans*P
>> ans*P
```

And so on. What do you observe?

**Question 71.3:** Is it still true that the predictions $\vec{\mathbf{x}}(t)$ for subsequent days approach a stationary distribution $\vec{\mathbf{x}}$?

**Question 71.4:** Based on your explorations, how would you estimate the proportions of rainy days and sunny days in a long sequence of days? How are these estimates related to (the) stationary distribution(s) of the corresponding Markov chain?