(IN)CONSISTENCY: SOME LOW-DIMENSIONAL EXAMPLES

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Abstract. This note is a slightly revised version of an earlier note with the same title that was part of a larger research project of the Dynamical Systems Group at Ohio University. While the major findings of the project are described in [1], this note complements [1] as it contains a more extensive review of some basic low-dimensional examples of Boolean systems and their ODE counterparts and explores whether the ODE dynamics is consistent with the Boolean dynamics.

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1. Notation and some basic definitions

Our notation roughly follows the one in [1], but as the original version of this note substantially predates [1], there are some important differences. Since the original version of this note forms part of a larger body of interconnected notes that contain additional unpublished results of independent interest, we will retain here the original terminology. Therefore let us briefly review some basic definitions and frequently used notations and display the differences between the terminology used here and the one of [1] in the form of remarks.

Throughout this note, we let f denote the updating function of an n-dimensional Boolean system. We will let $d(x, y)$ denote the Euclidean distance between vectors $x, y \in \mathbb{R}^n$.

We consider the following functions on the set of reals:

$$g(x_i) = 3x_i - x_i^3 - 3$$

$$S(x_i) = \begin{cases} 0 & \text{if } x_i \leq -1, \\ .5(x_i + 1) & \text{if } -1 < x_i < 1, \\ 1 & \text{if } x_i \geq 1 \ . \end{cases}$$

$$s(x_i) = \begin{cases} 0 & \text{if } x_i \leq 0, \\ 1 & \text{if } x_i > 0. \end{cases}$$

Remark: [1] *uses the notation* $L(x_i)$ *instead of* $S(x_i)$ *and* $S_i(\vec{x})$ *instead of* $s(x_i)$.

We need to be careful about using $s(\vec{x})$ and $S(\vec{x})$. When comparing an n-dimensional *Boolean network* with another system we let $s(\vec{x}) = (s(x_1), \dots s(x_n))$ regardless of whether $\vec{x}$ is in $\mathbb{R}^n$ or $\mathbb{R}^{2n}$. On the other hand, we will let $S(\vec{x}) = (S(x_1), \dots S(x_n))$ whenever the *ODE system* has n-dimensions.

We construct associated ODE systems $D_1(f, \vec{\gamma})$ and $D_2(f, \vec{\gamma})$ for any n-dimensional Boolean system $\mathbb{B}$ with updating function f. The system D_1 is defined in the following manner: for each $i \in [n]$ we let:

$$\dot{x_i} = \gamma_i(g(x_i) + 6P_i(S(\vec{x}))), \tag{1}$$

where $\gamma_i > 0$. One can think about the γ_is as constants, but our arguments will not be affected if the γ_is are allowed to depend on the state or even change over time, as long as they are all bounded and bounded away from zero, that is, if there are constants $M > m > 0$ such that $m < \gamma_i(\vec{x}, t) < M$ for all $i, \vec{x}$, and t.

We will consider (1) for real-valued function $P_i : \mathbb{R}^n \to \mathbb{R}$ that have some of the following five properties:

1 P_i takes the same values as f_i on vectors of zeros and ones.
2 P_i is continuous and maps $[0, 1]^n$ into $[0, 1]$.
3 P_i is a polynomial function.
4 P_i has the smallest possible degree.
5 P_i is *faithful,* which means that the sign of $P_i(\vec{x})$ can change only when at least one coordinate of $\vec{x}$ is zero.

Remark: *We don't require here that* P_i *has* all *of the above properties simultaneously, which may be impossible in any case for most higher-dimensional Boolean systems. The exposition in* [1] *goes one step further in that it considers more general continuous functions* Q_i *that play the same role as the functions* $P_i \circ S$ *of this note and satisfy a suitable generalization of Property 1.*

The definition of D_2 in some ways just reuses D_1 after modifying f. Let $f = (f_1, f_2, \dots, f_n) : 2^n \to 2^n$ be given, where 2^n is our shorthand for $\{0, 1\}^n$. Recall that f is the updating function for a uniquely determined n-dimensional Boolean system $\mathbb{B}$. We want to extend f to an updating function $f^+ : 2^{2n} \to 2^{2n}$ of a $2n$-dimensional Boolean system $\mathbb{B}^+$. Now for each $i \in [n]$ we define an auxiliary functions $c_i(\vec{s}) = s_{n+i}$ that copies the value of variable number $n + i$ to variable number i. Finally, let

$$f^+ = (c, f) = (c_1, \dots, c_n, f_1, \dots, f_n), \tag{2}$$

and define D_2 in the following manner: $D_2(f, \vec{\gamma}) = D_1(f^+, \vec{\gamma})$.

Let x^- be the unique root of the polynomial $g(x_i) = 3x_i - x_i^3 - 3$ and let x^+ be the unique root of the polynomial $g(x_i) + 6 = 3x_i - x_i^3 + 3$. Then $x^- \approx -2.1038$ and $x^+ \approx 2.1038$.

Lemma 1. *Let f define any n-dimensional Boolean system and let $\vec{\gamma}$ denote any vector of positive reals of suitable dimension. Then $[x^-, x^+]^n$ is a forward-invariant set in $D_1(f, \vec{\gamma})$ and $[x^-, x^+]^{2n}$ is a forward-invariant set in $D_2(f, \vec{\gamma})$.*

Proof. Let $\vec{x}(0) \in [x^-, x^+]^n$. If the trajectory $\phi_{\vec{x}}$ escapes $[x^-, x^+]^n$ then there exist a time τ and some variable x_i such that $x_i(\tau) \notin [x^-, x^+]$. Because our functions are continuous we know there must exist a time t such that $x_i(t) = x^-$ with a negative derivative or such that $x_i(t) = x^+$ with a positive derivative. Let's deal with the case that $x_i(t) = x^-$. Then equation (1) becomes:

$$\dot{x_i} = \gamma_i(3x_i - x_i^3 - 3 + 6P_i(S(\vec{x}))) = \gamma_i(6P_i(S(\vec{x}))) \tag{3}$$

But the sigmoid function S varies between zero and one, so we've seen that $0 \leq \dot{x_i} \leq 6\gamma_i$. In other words we can make it as fast or slow as we want but we can't make it negative. If x_i reaches x^- it will either be pushed back (perhaps slowly, or after a period of time) into the interval or x^- is a fixed point for the variable x_i.

A symmetric situation occurs if x_i tries to escape past x^+. Then equation (1) becomes:

$$\dot{x_i} = \gamma_i(3x_i - x_i^3 + 3 + 6P_i(S(\vec{x})) - 6) = \gamma_i(6P_i(S(\vec{x})) - 6). \tag{4}$$

But since $P_i(S(\vec{x})) \leq 1$, the right-hand side of (4) will never be positive, and we can argue as in the previous case. $\square$

These sets are not actually invariant, but this does not bother us, since we only care about forward trajectories and their Boolean counterparts anyway. Thus in view of Lemma 1 we will henceforth consider the state space of $D_1(f, \vec{\gamma})$ to be $[x^-, x^+]^n$ and the state space of $D_2(f, \vec{\gamma})$ to be $[x^-, x^+]^{2n}$. Note that both of these state spaces are compact and connected.

Assume that some ODE system

$$\dot{\vec{x}} = p(\vec{x}) \tag{5}$$

as above is given. For every ODE trajectory with initial condition $\vec{x}(0)$ we defined a symbolic real-time trajectory $\Psi(\vec{x}(0))$ on a time interval T by $\Psi(\vec{x}(T)) = \{s(\vec{x}(t)) : t \in T\}$. We may think of $\Psi(T)$ as the ODE implementation of a Boolean trajectory for initial condition $\vec{x}(0)$ on T. We will spend a good deal of time considering the "quality" of this implementation. Of particular importance for us will be how many times the ODE approximation of the Boolean model changes Boolean states on T.

Definition 1. *For an initial condition $\vec{x}(0)$ and a time interval T, we will say $\vec{x}(T)$ is* switchless *when $\Psi(\vec{x}(T)) = \{s\}$ for some $s \in 2^n$. In this case we will simply write $\Psi(\vec{x}(T)) = s$.*

Definition 2. *Let U be a subset of the state space of the ODE system (5).*

We will call p and f strongly consistent *on U when for every initial condition $\vec{x}(0) \in U$ there exists a sequence $(T_\tau)_{\tau \in \mathbb{N}}$ of pairwise disjoint consecutive and nondegenerate intervals with $\bigcup_{\tau \in \mathbb{N}} = [0, \infty)$ such that for all $\tau \in \mathbb{N}$*

(i) $\vec{x}(T_\tau)$ is switchless, and

(ii) The sequence $(\Psi(\vec{x}(T_\tau)) : \tau \in \mathbb{N})$ is a Boolean trajectory in $\mathbb{B} = (2^n, f)$, which means here that if $s(T_\tau) = s$, then either s is a fixed point of f and $s(T_\tau) = s(T_{\tau+1})$, or s is not a fixed point of f, and $s(T_{\tau+1}) = s^+ \neq s$, where $f_i(s) = s_i^+$ for all coordinates i with $s_i \neq s_i^+$.

If for all $\vec{x}(0) \in U$ the sequence $(\Psi(\vec{x}(T_\tau)) : \tau \in \mathbb{N})$ is what we call here a synch trajectory of $\mathbb{B} = (2^n, f)$, that is, if for all $\tau \in \mathbb{N}$ and $\vec{x}(0)$ we have

(iii) $\Psi(\vec{x}(T_\tau)) = f^\tau(s(\vec{x}(0)))$,

then we will call p and f strongly s-consistent *on U.*

Remark: *The notion of strong consistency that we are using here is equivalent to the notion of consistency as defined in* [1], *while the notion of strong s-consistency is equivalent to the notion of strong consistency in the sense of* [1]. *A weaker notion of consistency that had been investigated in this research project is neither considered here nor in* [1]. *Excluding the latter notion from our consideration allowed for the more streamlined equivalent definitions given in* [1].

The notion of "Boolean trajectory" as defined in Definition 2(ii) above is not explicitly used in [1]*; instead, the phrase as used in* [1] *refers to what is called "synch trajectory" in the above definition.*

For any given p and f there exist *maximal* $U = U(f, p)$ and $U^s = U^s(f, p)$ such that p and f are strongly (s-)consistent on U (U^s). These sets consist of all initial conditions $\vec{x}(0)$ for which the ODE trajectory is strongly (s-)consistent with the Boolean dynamics given by f.

In general, $U(f, p)$ may be a tiny subset of the state space or may even be empty. It is not immediately clear when we can assume $U(f, p)$ to be an open set or even to have nonempty interior. The next definition describes some additional desirable properties of $U(f, p)$ or $U^s(f, p)$.

Definition 3. *Let St denote the state space of (5), let $f : 2^n \to 2^n$, and let $U = U(f, p)$ or $U^s(f, p)$.*

(i) We say that U is complete *if for every Boolean state $s \in 2^n$ there exists an nonempty open $V \subset U$ with $s(\vec{x}) = s$ for every $\vec{x} \in V$.*

(ii) We say that U is universal *if the set $V := \{\vec{x} \in St : \exists t \geq 0 \ \vec{x}(t) \in U\}$ contains a dense open subset of St of full Lebesgue measure.*

Remark: *What we call here "complete U" is called "universal U" in* [1]. *The notion of "universal U" that we defined above is not considered in* [1].

The set V of Definition 3(ii) will be referred to as the set of initial conditions whose trajectories are *eventually (s-)consistent* with the Boolean dynamics. Thus $U(U^s)$ is universal if eventual (s)-consistency holds on almost the entire state space. Example 4 below shows that $U^s(f,p)$ may be complete without being universal and Proposition 7 below shows that $U^s(f,p)$ may be universal without being complete.

In the remainder of this note we will explore behavior of the notions that we reviewed above for some very simple low-dimensional examples of Boolean systems. We will start with the simplest possible Boolean systems and then work our way up to slightly more complicated ones.

2. $D_1(f,\gamma)$ for Boolean constants f in one dimension

Let $\mathbb{B} = (2, f)$ be a Boolean system of dimension one with a constant updating function. There are exactly two such systems, given by $f(s) \equiv 0$ and $f(s) \equiv 1$. We need only one variable with index $i = 1$ here and we have $n = 1$, but we will still write x_i, P_i, and $[0,1]^n$ in view of later work.

The most natural choices for P_i are the constant functions $P_i(\vec{x}) \equiv 0$ if $f(s) \equiv 0$ and $P_i(\vec{x}) \equiv 1$ if $f(s) \equiv 1$. Then we get s-consistency on the whole state space. In fact, this works under more general assumptions about P_i.

Proposition 2. *Assume $f : 2 \to 2$ is a constant Boolean function, $\gamma_i > 0$, and P_i satisfies Conditions 1 and 2 above is such that $P_i([0,1]^n) \subset [0,1/6)$ or $P_i([0,1]^n) \subset (5/6,1]$. Then (1) and f are strongly s-consistent on the whole state space $[x^-, x^+]^n$.*

Proof: Under our assumptions, the right-hand side of (1) has only one globally stable equilibrium x^* outside of the interval $[-1,1]$ at all times, with $x^* < -1$ if $f \equiv 0$ and $x^* > 1$ if $f \equiv 1$. Thus any trajectory in $D_1(f,\vec{\gamma})$ will move towards this equilibrium. It will cross the threshold of 0 at most once and the Boolean state $s(t)$ defined by

$$
\begin{aligned}
s(t) = 0 \quad &\text{if} \quad \vec{x}(t) < 0 \\
s(t) = 1 \quad &\text{if} \quad \vec{x}(t) \geq 0
\end{aligned}
\tag{6}
$$

will eventually be the fixed point of $\mathbb{B} = (2, f)$. Strong s-consistency immediately follows. $\square$

It may seem puzzling that Propositions 2 and 7 use the assumption that $P_i([0,1]^n) \subset [0,1/6)$ or $P_i([0,1]^n) \subset (5/6,1]$. If f_i is a Boolean constant, why would we want to use anything else for P_i than the corresponding constant polynomial? The answer is that we don't really *want* to use other P_is, but such alternatives may naturally result from our conversion methods. For example, the Boolean expression $s_1 \wedge s_1 \wedge \neg s_1$ is a contradiction and thus equivalent to the Boolean constant zero. The recursive conversion schemes $\mathcal{Q}_{Rc}$ and $\mathcal{Q}_{Rd}$ described in [1] translate it into the polynomial $P_1(x_1) = (x_1)^2(1 - x_1)$ which maps $[0, 1]$ onto $[0, 0.1481]$ and thus still satisfies the assumptions of Proposition 2. On the other hand, $(s_1 \wedge \neg s_1)$ also represents a contradiction, but it gets translated into a polynomial P_1 that takes all values in the interval $[0, 1/4]$ and thus does not satisfy the assumption of Proposition 2. We will return to this example below (Example 3).

On the other hand, the conversion method $\mathcal{Q}_W$ described in [3] always gives polynomials P_i of minimal degree, which for Boolean constants are necessarily constant, no matter how the tautology or contradiction is actually represented as a Boolean expression.

So perhaps we should simply adopt the conversion method $\mathcal{Q}_W$ instead of investigating possibly pathological interpretations of Boolean constants? This may not be a good idea, for three reasons.

First of all, for large Boolean systems the conversion method of $\mathcal{Q}_W$ requires a lot of time to compute; ours can be implemented in a much faster way.

Second, if we aim at results of largest possible generality, we also need to deal with P_i's that are in some ways less than optimal. Notice, for example, that the conversion of $s_1 \wedge s_1 \wedge \neg s_1$ into the polynomial $P_1(x_1) = (x_1)^2(1 - x_1)$ is quite natural, but not optimal in the above sense.

Third, we want to build up some results that we can use in a more general setting. Suppose for example that $f_1 = s_1 \wedge s_3$. Even the method $\mathcal{Q}_W$ will translate this into a quadratic polynomial. However, if we investigate the behavior of a trajectory along which $s(x_3) = 0$, then f_1 will behave along this trajectory as a Boolean constant in exactly the same way as any contradiction. The more general result Proposition 7 may give us a tool for investigating this trajectory, while a result with the more stringent assumption that P_i be constantly equal to zero wouldn't.

The following example shows that the assumptions of Proposition 2 can be weakened to some extent.

Example 3. *Let $f(s_1) = s_1 \wedge \neg s_1$. Then the corresponding polynomial $P_1(x_1) = (x_1)(1 - x_1)$ does not satisfy the assumptions of Proposition 2, but the corresponding ODE implementation $D_1(f, 1)$ is still strongly consistent with f on the whole state space.*

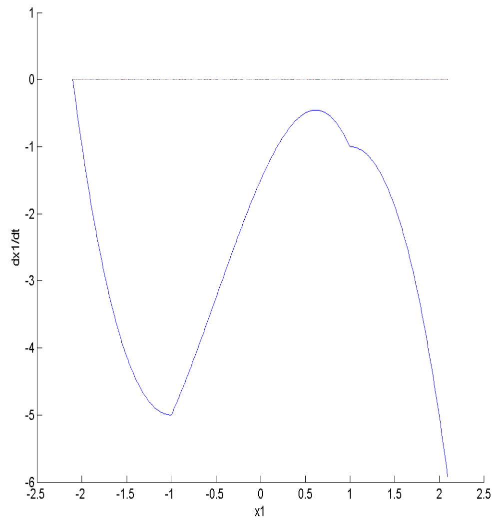
Proof: The ODE for the unique variable x_1 is

$$\dot{x_1} = 3x_1 - x_1^3 - 3 + 6S(x_1)(1 - S(x_1)). \tag{7}$$

We can get a feeling for this function by examining Figure 1.

We find that this cubic has only one zero at x^-, so this example gives $p(x)$ and $f(s)$ which are strongly consistent on the whole state space, with $\{x^-\}$ being the only attractor. The proof is identical to that of Proposition 2. □

FIGURE 1. $\dot{x_1} = 3x_1 - x_1^3 - 3 + 6S(x_1)(1 - S(x_1))$



However, some assumptions beyond 1 and 2 on P_i are necessary in Proposition 2.

Example 4. *Let $k = s_1 \wedge s_1 \wedge s_1 \wedge s_1$ and let $f_1 = k \wedge \neg k$. Then $U^s(f, p)$ for the ODE implementation $p = D_1(f, 1)$ based on the corresponding polynomial $P_1(x_1) = x_1^4(1 - x_1^4)$ is complete but not universal.*

Proof: The ODE for the unique variable x_1 is

$$\dot{x_1} = g(x_1) + 6S(x_1)^4(1 - S(x_1)^4). \tag{8}$$

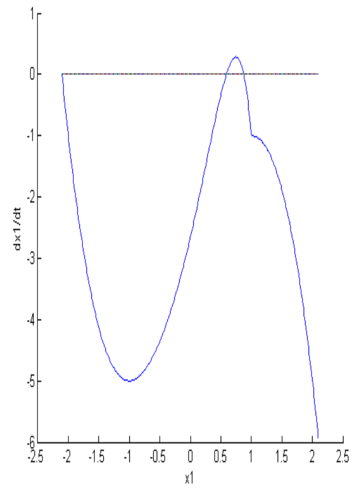
We can get a feeling for this function by examining Figure 2.

The system has three fixed points $r_1 = x^-, r_2 = .58875, r_3 = .87703$. Let us consider $x_1(0) \geq r_2$. Then there is no t such that $\Psi(x_1(t)) = 0$. This demonstrates that the system is not eventually consistent on any $U \subseteq [r_2, x^+)$. On the other hand, if we let $U = [x^-, r_2)$ we have strong s-consistency on U. Thus $U^s(f, p) = [x^-, r_2)$, which is complete but not universal. $\square$

The following example generalizes Example 4 and identifies the mechanism responsible for the observed dynamics.

Example 5. *Assume $f : 2 \to 2$ is the constant Boolean function $f \equiv 0$ and $\gamma_1 > 0$. Moreover, assume that P_1 satisfies Conditions 1 and 2 above is such that $g(x_1) + P_i(S(x_1)(0)) > 0$ for some $x_1(0)$ with $x_1(0) > 0$. Then the trajectory of $x_1(0)$ in (1) is not eventually strongly consistent with f.*

FIGURE 2. $\dot{x_1} = g(x_i) + 6S(x_1)^4(1 - S(x_1)^4)$



Proof: At time $t = 0$ there will be a locally stable equilibrium $x^*(0) > 1$ of (1) and $\dot{x}_1(x_1(0)) > 0$, so x_1 will move towards x^*. This situation will persist over some time interval T; for all $t \in T$, the variable $x_1(t)$ will increase and move towards a changing equilibrium $x^*(t) > 1$. In particular, $x_1(t)$ will not cross 0 as it should if the trajectory were consistent with f. In order for x_1 to change direction, $g(x_1) + P_i(S(x_1)(0))$ would need to become negative. But by the Intermediate Value Theorem, this would require $\dot{x}_1(t_1) = 0$ at a right endpoint t_1 of T, in which case the trajectory of $x_1(0)$ would reach a fixed point whose Boolean state $s(x_1(t_1)) = 1$ is inconsistent with f. If P_i is Lipschitz continuous rather than merely continuous, the trajectory of $x_1(0)$ will never actually reach a fixed point and T will be infinite. $\square$

Notice that the assumptions of Example 5 contradict the assumptions of Proposition 2, but they are not an outright negation of the latter.

Problem 1. *Formulate assumptions that are both necessary and sufficient in Proposition 2 and prove a versions of the proposition under these more general assumptions.*

Proposition 6. *For any ODE implementation $p = D_1(f, \gamma)$ of a contradiction or tautology $f : 2 \to 2$ the set $U^s(f, p)$ has nonempty interior.*

Proof: We prove the proposition for the case of a contradiction; the case of a tautology is analogous. Note that $\dot{x_1}(x^-) < 0$ by Condition 1 on P_1. Moreover, since P_1 is continuous by Condition 2, Thus there exists an $\epsilon > 0$ such that for all y with $|y - x^-| < \epsilon$ we have $\dot{x_1}(y) < 0$. It follows that $U = [x^-, \epsilon)$ is as required in the proposition. $\square$

3. $D_1(f, \vec{\gamma})$ for Boolean constants f in higher dimensions

Proposition 2 easily generalizes to the following result:

Proposition 7. *Assume $f : 2^n \to 2^n$ is a Boolean function such that each component f_i of f is a Boolean constant. Assume $\gamma_i > 0$ for all $i \in [n]$ and that each P_i satisfies Conditions 1 and 2 above and is such that $P_i([0,1]^n) \subset [0, 1/6)$ or $P_i([0,1]^n) \subset (5/6, 1]$. Then (1) and f are strongly consistent and eventually strongly s-consistent on the whole state space $[x^-, x^+]^n$. However, if $n > 1$, then the set on which (1) and f are strongly s-consistent is not complete.*

Proof: The proof of strong consistency is exactly the same as the proof of Proposition 2, since we can treat each variable separately. The last sentence will follow from Lemma 14 of the Appendix. We will defer its formal proof and instead give two illustrative examples here. □

For our first example, assume that s^* is the steady state of the Boolean system and $\vec{x}(0)$ is an initial state with $s(x_i(0)) = 1 - s^*_i$ and $s(x_j(0)) = 1 - s^*_j$ for some $i \neq j$, then both x_i will cross zero at some time $t_i > 0$ and x_j will cross zero at some time $t_j > 0$. For the trajectory of $\vec{x}(0)$ to be s-consistent with f, these crossings would have to happen at exactly the same time. This may be true for an individual $\vec{x}(0)$, but not for all initial conditions in an open neighborhood of $\vec{x}(0)$. To see why, consider the simplest case where $P_i = P_j$ are constant. Then the crossing times t_i and t_j depend monotonically on $x_i(0)$ and $x_j(0)$ in an identical fashion, and we will have $t_i = t_j$ only if $x_i(0) = x_j(0)$. Thus if U is the set of all initial conditions $\vec{x}(0)$ with $s(x_i(0)) = 1 - s^*_i$ and $s(x_j(0)) = 1 - s^*_j$ while $s(x_k(0)) = s^*_k$ for $k \in [n] \backslash \{i, j\}$, then $U^s(f, p) \cap U$ will be the nowhere dense subset of U that is obtained by intersecting U with the hyperplane $\{\vec{x} : \ x_i = x_j\}$.

For our second example, let us consider the two-dimensional f given by the following Boolean rules:

$$f_1(s) = \neg(s_1 \vee s_2) \vee s_1 \tag{9}$$

$$f_2(s) = \neg(s_1 \vee s_2) \vee s_2. \tag{10}$$

The functions above are not constant, but the example is still easy to analyze and nicely illustrates the phenomenon of non-simultaneous crossings, so we include it here. Note that $f = (f_1, f_2)$ maps each Boolean state $s \in \{01, 10, 11\}$ to itself, while 00 is mapped to 11. Our standard conversion method to $D_1(f, \vec{1})$ gives us the following set of equations:

$$\begin{aligned} P_1(S(x)) &= (1 - S(x_1)(1 - S(x_2)) + S(x_1) - (1 - S(x_1)(1 - S(x_2))S(x_1) \\ P_2(S(x)) &= (1 - S(x_1)(1 - S(x_2)) + S(x_2) - (1 - S(x_1)(1 - S(x_2))S(x_2). \end{aligned} \tag{11}$$

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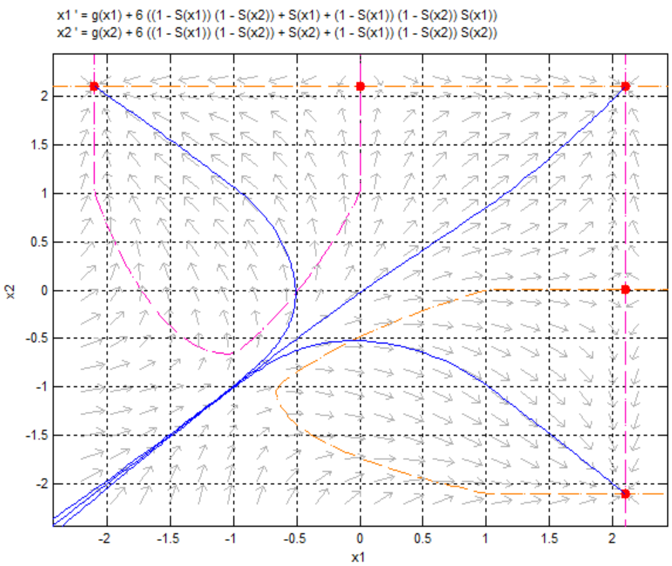
FIGURE 3. Phase portrait of the D_1 counterpart of a Boolean network where every element is a fixed point except 00 which is succeeded by 11.



This results in a phase portrait given by Figure 3 which illustrates a number of things. First of all, one can see that from many initial conditions, the ODE system will approach a steady state that corresponds to the correct Boolean steady state of f. However, for initial conditions with $x_1(0), x_2(0) < 0$, the ODE dynamics will be strongly s-consistent with the Boolean dynamics only if $x_1(0) = x_2(0)$, similarly to the situation in the previous examples. Moreover, for all the initial conditions below or to the left of the two curved sample trajectories shown, the system will approach one of the two steady states (x^+, x^-) or (x^-, x^+). Since these regions include many initial conditions with $x_1(0), x_2(0) < 0$, we will not even have consistency for most trajectories starting with $x_1(0), x_2(0) < 0$. Finally, we note that the ODE system also has two unstable fixed points at $(0, x^+)$ and $(x^+, 0)$ which do not have Boolean counterparts.

Again, the assumptions of Proposition 7 are stronger than necessary.

Example 8. *Consider the following two-dimensional Boolean Network: Any initial condition* (s_1, s_2) *is succeeded by* $(0, 0)$. *We let*

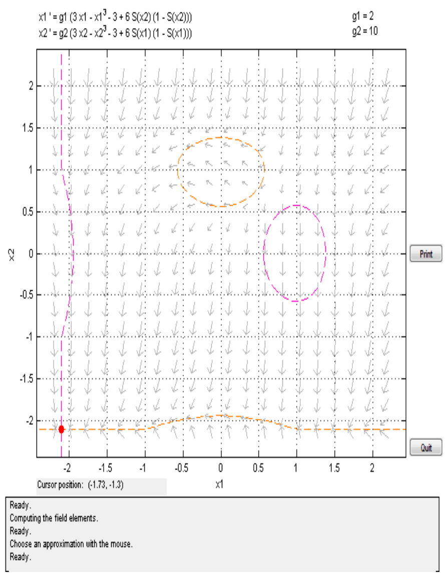
$$f_1(s) = s_2 \wedge \neg s_2 \qquad f_2(s) = s_1 \wedge \neg s_1. \tag{12}$$

Letting $\dot{x}_i = g(x_i) + 6P_i(S(\vec{x}))$ *as prescribed by* $D_1(f, \vec{\gamma})$ *with the standard implementations of the* P_i*s we find:*

$$\begin{aligned} \dot{x}_1 &= \gamma_1[g(x_1) + 6S(x_2)(1 - S(x_2))] \\ \dot{x}_2 &= \gamma_2[g(x_2) + 6S(x_1)(1 - S(x_1))] \end{aligned} \tag{13}$$

Taking $\gamma_1 = 2$ and $\gamma_2 = 10$ gives us the phase portrait seen in Figure 4.

FIGURE 4. Phase Portrait of (13).



The part of the x_1-nullcline centered at $(0, 1)$ and the part of the x_2-nullcline centered at $(1, 0)$ present no serious problem and the only attractor of this system is given by $\{(x^-, x^-)\}$. This proves that again this system and $f(s)$ are strongly consistent on $U = [x^-, x^+]^2$.

It's important to note that our selection of $(\gamma_1, \gamma_2) = (2, 10)$ had no impact on the location of nullclines. But if we warp these nullclines we can introduce additional fixed points.

Example 9. *Consider the following two-dimensional Boolean network: Any initial condition (s_1, s_2) is succeeded by $(0, 0)$. Let $k = s_1 \wedge s_1 \wedge s_2 \wedge s_2$ and*

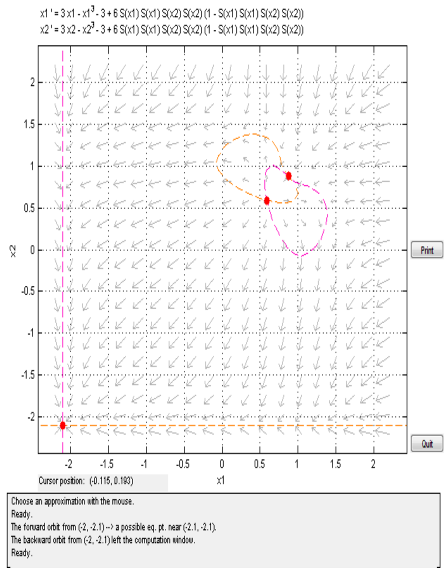
$$f_1(s) = k \wedge \neg k \qquad f_2(s) = k \wedge \neg k. \tag{14}$$

Construct $D_1(f, \vec{\gamma})$ by taking $\vec{\gamma} = (1, 1)$, letting $k^* = (S(x_1)S(x_2))^3$ and using the standard ODE implementation $p = D_1(f, (1, 1))$ of (14)

$$\dot{x}_i = g(x_i) + 6k^*(1 - k^*). \tag{15}$$

for $i \in \{1, 2\}$.

FIGURE 5. Phase Portrait of (15).



This system has 3 fixed points: (x^-, x^-), $(.58875, .58875)$, and $(.87703, .87703)$. the phase portrait can be seen in Figure 5.

Because for this system $x_1 = x_2$ implies $\dot{x}_1 = \dot{x}_2$ we find that $Y = \{(y, y) \in [x^- x^+]^2\}$ is invariant. This shows us that this system isn't strongly consistent on $[x^-, x^+]^2$. Consider the initial condition $\vec{x}(0) = (2, 2)$. If the system of ODEs in this example and $f(s)$ were strongly consistent there there would exist at least one $t > 0$ such that $\Psi(x(t)) = f(\Psi(x(0))) = 0$. This tells us that $x(t)$ has two non-positive components. But because Y is invariant this means that the trajectory would have to pass through $(.87703, .87703)$ which is impossible as this is a fixed point. However, we can see that f and p are strongly consistent on $[x^-, x^+]^2 \backslash Y$; in other words, $[x^-, x^+]^2 \backslash Y \subseteq U(f, s)$. Thus $U(f, s)$ is complete and universal. One can also easily see that the intersection of the set $U^s(f, p)$ with the first quadrant is contained in Y, so that $U^s(f, s)$ is universal but not complete. The latter was to be expected from our earlier observations about nongenericity of simultaneous crossings of boundaries.

It is quite interesting to note that in contrast with Example 4 we do get a universal $U^s(f, p)$. This cannot happen in one dimension, where for constant f we must have either $U^s(f, p) = [x^-, x^+]$ or $U^s(f, p)$ note universal. The additional dimension provides an opportunity for trajectories to move around the problematic areas. We will make good use of this effect in our later work.

Problem 2. *(a) Formulate assumptions that are both necessary and sufficient in Proposition 7 and prove a version of the proposition under these more general assumptions.*

(b) Formulate assumptions that are both necessary and sufficient in Proposition 7 if we replace "eventually strongly s-consistent on the whole state space $[x^-, x^+]^n$" by "$U^s(f,p)$ is universal" in its conclusion and prove a version of the proposition under these more general assumptions.

Part (a) of Problem 2 should not be too difficult; part (b) is more interesting, but also likely to be more challenging.

4. A generalization: $D_1(f,\vec{\gamma})$ for loop-free f

Recall that with any n-dimensional Boolean system with updating function f we can associate a directed graph $D_f = ([n], A_f)$ called the *connectivity of* f such that $< j, i > \in A_f$ iff variable s_j acts as an essential input in the regulatory function f_i. We will write D instead of D_f and A instead of A_f if f is implied by the context. We call f *loop-free* if D contains no directed cycles. Boolean constants as in the previous section are loop-free. The simplest examples of Boolean functions f that are not loop-free have dimension 1 and $A = \{< 1, 1 >\}$.

In any loop-free Boolean system the set of nodes $[n]$ can be partitioned into levels; $[n] = \bigcup_{\xi=0}^{\kappa} L_\xi$, where

- $L_0 \neq \emptyset$ and L_0 consists of all variables with constant regulatory functions; that is, of all variables with indegree 0 in D.
- $L_{\eta+1}$ consists of all variables i such that $i \notin \bigcup_{\xi=0}^{\eta} L_\xi$ and $j \in \bigcup_{\xi=0}^{\eta} L_\xi$ for all $< j, i > \in A$.

Consider the sync trajectory of initial state $s(0)$ for a loop-free f. For all $i \in L_0$, the Boolean state $s_i(\tau)$ remains constant for all $\tau \geq 1$. Variables in L_1 take their inputs only from variables in L_0, so $s_i(\tau)$ will remain fixed for all $\tau \geq 2$. By induction it follows that the system will reach a unique steady state s^* after at most $\kappa + 1$ steps. (Notice that in our treatment of "Boolean constants" these variables need to take their constant state only for times $t \geq 1$).

Lemma 10. *Let $f : 2^n \to 2^n$ be loop-free and let $\vec{\gamma}$ be an n-dimensional vector of positive reals. Assume that for all i with $s_i \in L_0$ the assumptions of Proposition 7 are satisfied by P_i, and assumptions 1 and 2 are satisfied by all P_i. The the dynamics of $D_1(f,\vec{\gamma})$ is eventually strongly consistent on the whole state space with f.*

The proof of Lemma 10 is left as an exercise.

The really interesting Boolean systems are not loop-free. Therefore, Lemma 10 is of somewhat limited interest all by itself. However, the lemma fails to generalize in some illuminating ways, which may help us build up some helpful intuitions for the later parts of our project.

5. $D_1(f,\gamma)$ for nonconstant f in one dimension

When $n = 1$, then there are four Boolean systems of dimension n: Two of them represent Boolean constants. These were already dealt with in

Section 2. The other two have regulatory functions $f_c(s) = s_1$ (the "copy" function), and $f_{cn}(s) = 1 - s_1$ ("copy-negation"). Neither of the latter is loop-free. Let us take a closer look at these systems.

5.1. **The case $D_1(f_c, \gamma)$.** First consider our standard implementation $P_1(x_1) = x_1$. Then P_1 is faithful and $P_1 \circ S$ is piecewise linear. It follows that $D_1(f_c, \gamma)$ has three steady states: Locally asymptotically stable ones at x^-, x^+ and an unstable one at zero. For $x(0) < 0$ the trajectory will move towards x^-, for $x(0) > 0$ the system will move towards x^+, and for $x(0) = 0$ the trajectory will remain at the unstable fixed point. The exact same observation holds for every faithful P_1. We get the following.

Proposition 11. *Let $p = D_1(f_c, \gamma)$ be implemented by a faithful P_1. Then* $U^s(f, p) = [x^-, x^+] \backslash \{0\}$.

The assumption that P_1 be faithful is necessary in Proposition 11. For example, consider P_1 such that $P_1 \circ S(x)$ takes negative values for $x < x^c < 0$ and positive values for $x > x_c$. Then an inspection of the phase-line diagram of $p = D_1(f_c, \gamma)$ reveals that for $x^c < x(0) < 0$ the real-time Boolean trajectory $\Psi(x(0))$ will not be switchless, which is inconsistent with the Boolean dynamics. For such a choice of P_1 we still have eventual strong consistency on $[x^-, x^+] \backslash \{x^c\}$ though, which implies that $U^s(f, p)$ is universal.

We leave it as an exercise to construct an concrete example of P_1 that satisfies conditions 1 and 2 such that for the corresponding $p = D_1(f_c, \gamma)$ the set U^s is not universal.

5.2. **The case $D_1(f_{cn}, \gamma)$.** In this case, all Boolean trajectories satisfy

$$\ldots \mapsto 0 \mapsto 1 \mapsto 0 \mapsto 1 \mapsto \ldots$$

Consider our standard implementation of $p = D_1(f_{cn}, \gamma)$ with $P_1(x_1) = 1 - x_1$ and $P_1 \circ S$ is piecewise linear. Thus for $x < 0$ the form of (1) implies that $\frac{dx}{dt} > 0$, and for $x > 0$ the form of (1) implies that $\frac{dx}{dt} < 0$. We conclude that $D_1(f_{cn}, \gamma)$ has exactly one globally asymptotically stable steady state at zero. For any $x(0) \in [x^-, x^+]$ the trajectory of $x(0)$ in $D_1(f_{cn}, \gamma)$ will retain the Boolean state $s(x(0))$ at all times, and $U(f, s) = \emptyset$. Thus $D_1(f_{cn}, \gamma)$ will be maximally inconsistent with the Boolean dynamics of f_{cn}.

6. The case $n = 2$

In this case there exist already $2^8 = 256$ different Boolean systems of dimension n. Some of these are loop-free and covered by Section 4; some of them reversible; most are neither. Some of the reversible Boolean systems of dimension $n = 2$ are chaotic in the sense of the Derrida curve. For example, the system given by

$$00 \mapsto 00 \quad 01 \mapsto 11 \quad 10 \mapsto 01 \quad 11 \mapsto 10$$

has this property. Since $D_1(f, \vec{\gamma})$ is a two-dimensional ODE system for the latter, the ODE dynamics must be ordered. This gives, even prior to any simulations, the following result.

Corollary 12. *Chaos in a loop-free Boolean system does not imply chaos in the corresponding ODE system* $D_1(f, \vec{\gamma})$.

This is quite remarkable, but the absence of chaos in $D_1(f, \vec{\gamma})$ may be a bit of an artifact due to its low dimension. We still need to explore whether such examples exist, or even are the norm, in higher dimensions or if we work with other ODE analogues, such as $D_2(f, \vec{\gamma})$.

6.1. **The case $D_2(f_{cn}, \gamma)$.** Define a 2-dimensional updating function $f^+_{cn}(f_1, f_2) : 2^2 \to 2^2$ by choosing the following regulatory functions.

$$f_1(s) = 1 - s_2 \qquad f_2(s) = s_1. \tag{16}$$

This system is critical and reversible. A non-steady state attractor is given by

$$00 \mapsto 10 \mapsto 11 \mapsto 01 \mapsto 00, \tag{17}$$

and since this attractor comprises the whole state space of the Boolean system generated by f^+, it is the only one.

Define $p = D_1(f^+_{cn}, \vec{\gamma})$ by choosing $P_1 = 1 - x_2$ and $P_2 = x_1$ as in Section 5.

Now let us take a closer look: $D_1(f^+_{cn}, \vec{\gamma})$ is really nothing else but $D_2(f_{cn}, \vec{\gamma})$ with the roles of variables reversed. The reversal is a minor notational blunder, but the systems are clearly conjugate, so we leave our notation here as is in order to minimize the amount of necessary revisions. We found that for $D_1(f_c, \gamma)$ we cannot get any consistency between ODE and Boolean trajectories whatsoever. In a sense, we added just one dummy variable to $D_1(f_c, \gamma)$, and bingo! As we will show here, the resulting ODE system shows as much consistency with the Boolean dynamics as one could possibly hope for.

Let us be careful though that we are not getting ahead of ourselves here. Recall that with a state $\vec{x} = (x_1, \ldots, x_{2n})$ in the $2n$-dimensional state space of $D_2(f, \vec{\gamma})$ we associate a Boolean state $s(\vec{x}) = (s(x_1), \ldots, s(x_n))$ of dimension n only, that is, we ignore the auxiliary variables $x_{n+1}, \ldots, x_{2n}$. Then we construct a Boolean sequence $s^{\bar{t}}$ based on these n-dimensional vectors only, and hope that it will be a Boolean trajectory.

It is true for any n-dimensional Boolean system given by f that we can treat $D_2(f, \vec{\gamma})$ as $D_1(f^+, \vec{\gamma})$, but the correspondence between (sync) trajectories of f and f^+ is not straightforward. In the example discussed here such a direct correspondence does hold, but in Subsection 6.2 below we will give an example where sync trajectories of f^+ correspond to trajectories of f, but not to sync trajectories of f. In general not every trajectory of f^+ will correspond to a trajectory of f. Such a correspondence does hold for sync trajectories though. We will explore this issue in more detail in a subsequent note.

Since $n = 2$, we have the luxury of being able to perform an easy phase-plane analysis of $D_1(f^+_{cn}, \vec{\gamma})$. Figure 6 gives the phase portrait for the choice of parameters $\gamma_1 = \gamma_2 = 1$.

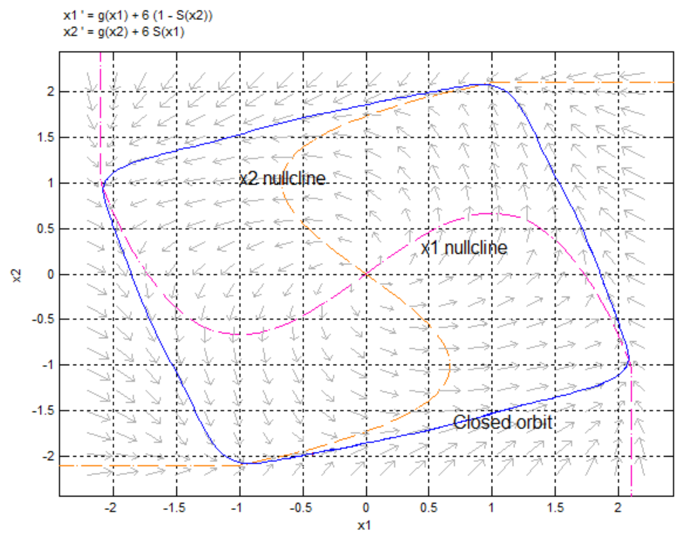


FIGURE 6. Nullclines and direction arrows for $D_1(f^+_{cn}, (1, 1))$.

The horizontal and vertical parts of the nullclines occur in the regions of the phase plane where $P_1(S(x_1, x_2))$ or $P_1(S(x_1, x_2))$ are constant. The most important fact we can learn from Figure 6 is:

The two nullclines intersect at $(0, 0)$, *which is the only steady state.*

Let us study this system analytically. For most of the conversion schemes described in [1] we have:

$$
\begin{array}{rclcc}
P_1(S(x_1,x_2)) & = & 1 & \text{for} & x_2 \leq -1 \\
P_1(S(x_1,x_2)) & = & 1 - 0.5(x_2+1) & \text{for} & -1 < x_2 < 1 \\
P_1(S(x_1,x_2)) & = & 0 & \text{for} & x_2 \geq 1 \\
P_2(S(x_1,x_2)) & = & 0 & \text{for} & x_1 \leq -1 \\
P_2(S(x_1,x_2)) & = & 0.5(x_1+1) & \text{for} & -1 < x_1 < 1 \\
P_2(S(x_1,x_2)) & = & 1 & \text{for} & x_1 \geq 1.
\end{array} \tag{18}
$$

In view of (1) and (16), the Jacobian at $(0, 0)$ is given by

$$
J = \begin{bmatrix} 3\gamma_1 & 3\gamma_1 \\ -3\gamma_2 & 3\gamma_2 \end{bmatrix} \tag{19}
$$

with eigenvalues

$$\lambda_1 = 1.5(\gamma_1 + \gamma_2) + 0.5\sqrt{9(\gamma_1 + \gamma_2)^2 - 72\gamma_1\gamma_2}$$
$$\lambda_2 = 1.5(\gamma_1 + \gamma_2) + 0.5\sqrt{9(\gamma_1 + \gamma_2)^2 - 72\gamma_1\gamma_2}. \tag{20}$$

Since $\gamma_1, \gamma_2 > 0$, we get two conjugate complex eigenvalues. Moreover, $1.5(\gamma_1 + \gamma_2) > 0$, and it follows that $(0,0)$ is an unstable focus. By the Poincaré-Bendixson Theorem, each trajectory that starts off the equilibrium $(0,0)$ will approach a limit cycle, and Figure 6 indicates that the ODE dynamics on $[x^-, x^+]^2 \backslash \{0,0\}$ will be strongly-s-consistent with the Boolean dynamics of f^+_{cn}. In other words, for $p = D_1(f^+_{cn}, \vec{\gamma})$ we have $U^s(f^+_{cn}, p) = [x^-, x^+]^2 \backslash \{0,0\}$, which is complete and universal. For this set of initial conditions, the Boolean trajectory of x_2 (which is the state variable of $D_2(f_{cn}, \vec{\gamma})$ under our reversed notation) will be

$$s^{\bar{t}} = (\dots, 0, 1, 0, 1, \dots),$$

which is exactly the dynamics of f_{cn}.

Inspection of Figure 6 reveals that the limit cycle visits a "clean state" for every Boolean state along the trajectory. This feature occurs in much more general situations and forms the basis for the main theorem in [1]. Moreover, notice that there exists exactly one limit cycle. This feature depends on the particular form of the P_i's which were chosen as the simplest possible ones. They are faithful polynomials of lowest possible degrees. In general, if we only assume the the P_i's satisfy conditions 1 and 2, the phase portrait may be more complicated and the set $U^s(f^+, p)$ does not need to be universal.

Problem 3. *Construct a specific example that confirms the claim made in the previous sentence.*

However, in view of the results in [1], the set $U^s(f^+, p)$ will always be complete, and will contain sufficiently clean states for every Boolean state.

Another interesting feature of this example is that we did not need to assume any separation of time scales. This contrasts with our work in [1] where such separation of time scales was assumed.

Problem 4. *For which systems is an assumption about separation of time scales actually needed to prove some version of consistency?*

6.2. **The case $D_2(f_c, \gamma)$.** Define a 2-dimensional updating function $f^+_c(f_1, f_2) : 2^2 \to 2^2$ by choosing the following regulatory functions.

$$f_1(s) = s_2 \qquad f_2(s) = s_1. \tag{21}$$

This system is critical and reversible, has two steady states 00, 11 and an attractor of length 2 that comprises the other two states 10 and 01.

Define $p = D_1(f^+_c, \vec{\gamma})$ by choosing $P_1 = x_2$ and $P_2 = x_1$ as in Section 5.

Figure 7 gives the phase portrait for $\gamma_1 = \gamma_2 = 1$.

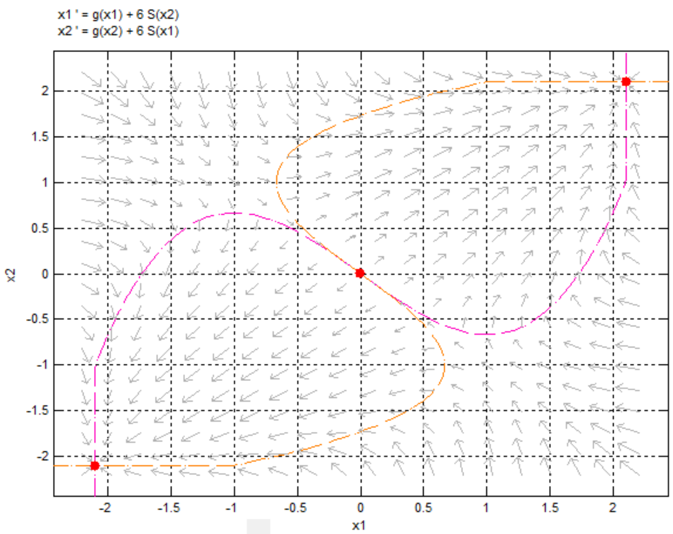


FIGURE 7. Nullclines and direction arrows for $D_1(f_c^+, (1,1))$.

Inspection of Figure 7 reveals that $(0,0)$ is the unique steady state and all trajectories that start outside a diagonal separatrix will approach one of the steady states (x^-, x^-), (x^+, x^+) that correspond to the Boolean steady states 00 and 11. By symmetry of the expressions for $\dot{x_1}, \dot{x_2}$, the separatrix Sep must consist of all states (x_1, x_2) such that $x_2 = -x_1$, and inspection of Figure 7 reveals that this separatrix is also the stable manifold of the equilibrium $(0,0)$. Let

$$U_= = \{\vec{x} \in [x^-, x^+]^2 : \ (x_1 x_2 > 0)\} \quad \text{and}$$

$$U_{\neq} = \{\vec{x} \in [x^-, x^+]^2 : \ (x_1 x_2 < 0\}.$$

One can also see from the phase portrait that $U_= \subset U^s(f,p)$ and that all trajectories that start outside of $[x^-, x^+]^2 \backslash Sep$ will eventually enter $U_=$. Thus $U^s(f,p)$ is universal.

But $U^s(f,p)$ is not complete; for completeness, $U^s(f,p)$ would need to contain some points from $U_{\neq}$. But inspection of Figure 7 also reveals that trajectories that start in $U_{\neq}$ either stay on Sep (in which case $\Psi(\vec{x})$ remains switchless) or will enter $U_=$, so that $\Psi(\vec{x})$ will contain exactly one Boolean switch. In this case the Boolean sequence will still be a trajectory of f_c^+, but not the sync trajectory. It follows that $U(f,s) = [x^-, x^+]^2 \backslash Sep$. Thus $U(f,s)$ is both complete and universal. In other words, the dynamics of p will be strongly consistent, but not strongly s-consistent with the Boolean dynamics of f_c^+ on $[x^-, x^+]^2 \backslash Sep$.

Now let us take a closer look: $D_1(f_c^+, \vec{\gamma})$ is really nothing else but $D_2(f_c, \vec{\gamma})$. From the point of view of f_c, the set $U_=$ contains representatives of every

Boolean state, and this set should be considered complete from this point of view. Thus for $D_2(f_c, \vec{\gamma})$ we get strong s-consistency on a complete subset of the state space.

7. An ODE system without periodic orbits

Let $n = 4$ and define a Boolean updating function $f : 2^4 \to 2^4$ by choosing the following regulatory functions.

$$
\begin{aligned}
f_1(s) &= 1 - s_2 \qquad & f_2(s) &= s_1, \\
f_3(s) &= 1 - s_4 \qquad & f_4(s) &= s_3.
\end{aligned} \tag{22}
$$

This system is critical and reversible. It is really nothing else than the direct product of the Boolean system defined by f_{cn} of Subsection 6.1 with itself. There are four disjoint attractors of length four each in this system:

$$
\begin{aligned}
&0000 \mapsto 1010 \mapsto 1111 \mapsto 0101 \mapsto 0000, \\
&0010 \mapsto 1011 \mapsto 1101 \mapsto 0100 \mapsto 0010, \\
&0011 \mapsto 1001 \mapsto 1100 \mapsto 0110 \mapsto 0011, \\
&0001 \mapsto 1000 \mapsto 1110 \mapsto 0111 \mapsto 0001,
\end{aligned} \tag{23}
$$

and their union is the whole state space. Note that these sync trajectories correspond to the attractor of f_{cn} given by (17). They differ by how far out of step the variables s_1, s_2 are with the variables s_3, s_4.

Now define $p = D_1(f, \gamma)$ analogously to the definition in Subsection 6.1. It follows from our previous work that the projection of almost any trajectory of $D_1(f, \vec{\gamma})$ on the (x_1, x_2)-plane approaches a stable limit cycle $\mathbf{C}_1$, while the projection on the (x_3, x_4)-plane approaches a stable limit cycle $\mathbf{C}_2$. Let us assume for simplicity that $\gamma_1 = \gamma_2 = 1$ and $\gamma_3 = \gamma_4$.

Then the minimal time T it takes for $(x_1(t), x_2(t)) \in \mathbf{C}_1$ to return to itself is fixed, while the minimal time $T(\gamma_3)$ it takes for $(x_3(t), x_4(t)) \in \mathbf{C}_2$ to return to itself depends continuously on γ_3. Thus the dynamics on the restriction of the state space of $D_1(f, \vec{\gamma})$ to $\mathbf{C} = \mathbf{C}_1 \times \mathbf{C}_2$ is topologically equivalent (even diffeomorphic, but we don't need this here) to the dynamics on a torus given by two maps on the unit circle defined by $\varphi_t(\beta) = \beta + \alpha t$ and $\psi_t(\beta) = \beta + \alpha(\gamma_3)t$, where $\alpha = \frac{2\pi}{T}$ and $\alpha(\gamma_3) = \frac{2\pi}{T(\gamma_3)}$, and we consider angles that differ by a multiple of 2π as equal. It is well known that if $\frac{\alpha}{\alpha(\gamma_3)}$ is irrational, then the latter dynamics is transitive (see [2], pp. 245/246). It follows that for most choices of γ_3 (actually, for most choices of $\vec{\gamma}$) the system $D_1(f, \vec{\gamma})$ does not have periodic orbits. However, this system does not have sensitive dependence on initial conditions; it is an example of a *quasi-periodic* system. These observations lead to the following result, whose formal proof is left as an exercise.

Proposition 13. *For the system defined above we have*

$$U(f,p) = [x^-, x^+]^4 \backslash \{\vec{x} : \; x_1 = x_2 = 0 \; \vee \; x_3 = x_4\}$$

regardless of the choice of $\vec{\gamma}$, but $U^s(f,p) = \emptyset$ except $\vec{\gamma}$ in a residual subset of $(0, \infty)^4$.

Thus in this example, strong consistency between the ODE and Boolean trajectories is a generic property, but strong s-consistency occurs only for very special choices of $\vec{\gamma}$.

The example in this section may appear largely irrelevant, since there is no interaction between variables in the set $\{x_1, x_2\}$ and variables in the set $\{x_3, x_4\}$; the system is *decomposable.* However, this type of dynamics may occur along trajectories in larger systems that are not decomposable. For example, this will happen when some other variables that mediate interactions between these sets take fixed values along the trajectories in question. It will also happen if the variables x_1, x_2, x_3, x_4 send input to other variables, but do not themselves receive input from other parts of the system.

8. Appendix: Nongenericity of strong s-consistency

Here we present the proof of a well-known general result in the theory of ODEs that precludes strong s-consistency on a complete subset of the state space for ODE implementations of most Boolean systems of interest.

Definition 4. *Let $f : 2^n \to 2^n$ be a Boolean updating function and let $D(f)$ be an ODE implementation of the corresponding Boolean system. We say that $D(f)$ is a* topologically nondegenerate implementation of f *if*

1. *The state space St of $D(f)$ is a compact m-dimensional topological manifold with boundary for some $m \geq n$.*
2. *The right-hand side of $D(f)$ is Lipschitz-continuous.*
3. *There are subsets $Z_1, \dots, Z_n \subset St$ such that for all $i \in [n]$ both Z_i and $St \backslash Z_i$ are m-dimensional topological manifolds.*
4. *For all $i \in [n]$ the boundary N_i of Z_i in St is a union of finitely many $m-1$-dimensional topological manifolds.*
5. *For all $i, j \in [n]$ with $i \neq j$ the intersection $N_i \cap N_j$ is a union of finitely many compact topological manifolds of dimensions $\leq m-2$.*
6. *The Boolean state $s_i(\vec{x})$ for $\vec{x} \in St$ will be interpreted as zero if $\vec{x} \in Z_i$ and as one if $\vec{x} \in Z_1$.*

Note that we do not require any smoothness conditions on the manifolds in Definition 4. For this reason we call the implementation "topologically" degenerate. In some subsequent results, we may need to impose more stringent conditions on the boundaries of the Z_is and it seems prudent to reserve the unmodified adjective "nondegenerate" for such purposes. In this definition we also do not require any kind of consistency between the ODE and the Boolean system; it suffices that we can define real-time Boolean trajectories.

The following lemma implies that for nondegenerate ODE implementations of Boolean systems the set of initial conditions whose trajectories cross multiple boundaries simultaneously is negligible.

Lemma 14. *Suppose $D(f)$ is a topologically nondegenerate ODE implementation of a Boolean system. Let $i \neq j$, and suppose that $\vec{x}(0)$ is an initial condition and $0 < t_0 < t_1$ are times with $\{\vec{x}(t) : t \in [0, t_1]\}$ contained in the interior of St such that*

(i) $\vec{x}(t_0) \in N_i \cap N_j$.

(ii) For all $\vec{y}(0)$ in some neighborhood U of $\vec{x}(0)$ we have

$$|\{t \in [0, t_1] : \vec{y}(t) \in N_i \cap N_j\}| \leq 1.$$

Then there exists a neighborhood V of $\vec{x}(0)$ such that the set

$$NS(i, j) = \{\vec{y}(0) : \forall t \in [0, t_1]\ \vec{y}(t) \notin N_i \cap N_j\} \tag{24}$$

contains a dense open subset of V.

Notice that condition (i) covers both the case when the Boolean states s_i, s_j change simultaneously at time t_0 and the case where the trajectory reaches the two boundaries at time t_0 and then turns back, as well as mixed scenarios. Condition (ii) precludes, among other things, trajectories that move along $N_i \cap N_j$ for a while. For all ODE implementations of Boolean systems of interest to us, condition (ii) will be satisfied on a dense open subset of the state space.

Proof of Lemma 14: Let everything in sight be as in the assumptions, and let W be a closed neighborhood of $\vec{x}(t_0)$. Define a map $F : W \times [0, t_1] \to St \times [0, t_1]$ by $F(\vec{z}(0), t) = (\vec{z}(t - t_0), t)$. This definition requires that we can extend ODE trajectories backwards in time, which may not always be the case (see Lemma 1 where we have only forward-invariance for our state space), but since we assumed that $\vec{x}(0)$ is in the interior of St we can choose W sufficiently small so that the relevant trajectories don't leave St in the time interval $[-t_1 + t_0, 0]$. Let K be the range of F.

By Theorem 3.16 of [2], F is continuous in both variables. Since $W \times [0, t_1]$ is compact, F is a homeomorphism between $W \times [0, t_1]$ and K. Thus K is a topological manifold of dimension $m + 1$. Let $V = \{\vec{x} \in St : (\vec{x}, t_0) \in K\}$. Thus V is the set of points whose trajectory resides in W at time t_0. This set is a neighborhood of $\vec{x}_0$ by continuity of F. Wlog (by choosing W sufficiently small) we can assume that $V \subset U$, where U is as in (ii). By condition (5) of Definition 4, $F(N_i \cap N_j)$ is a union of finitely many submanifolds of dimension $m - 1$ of K. Now let

$$V^* = \{\vec{y}(0) \in V : \exists t \in [0, t_1]\ \vec{y}(t) \in N_i \cap N_j\}.$$

Notice that V^* is the projection of $F(N_i \cap N_j)$ onto V. The projection map is continuous, and condition (ii) implies that its restriction to the compact set $F(N_i \cap N_j)$ is injective. Thus V^* is homeomorphic to $F(N_i \cap N_j)$ and thus is a union of finitely many manifolds of dimension $m - 1$. Since $int(V)$ has dimension m, the lemma follows. □

References

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