Mathotopia: An Old Fairy Tale, Retold in Mathematical Words

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True or false?

1. **We** mathematicians are good and rational people.
2. Irrational people look down upon us, ridicule us, frown upon us, discriminate against us, persecute us.
3. **They** cause **all** the trouble in the world.
4. If only we could keep **them** out of **our** world, eternal bliss and happiness would ensue.

Look familiar? Where does this get problematic?

1. There is nothing wrong with taking pride in one's profession, nationality, political views, or religion.
2. Most groups of people have at times been mistreated by other groups.
3. But the universal quantifier **"all"** puts us on a slippery slope
4. towards total disaster rather than eternal bliss.

At least **we mathematicians** can recognize that all versions of the above reasoning are **isomorphic** and equally irrational.

Let's dream the dream of mathotopia, an island where everybody is rational and gets along well with everybody else.

There may be upward of one billion rational people on this planet, and we might expect a large influx of would-be immigrants.

How should we handle this influx while keeping the promise of mathotopia alive?

How many of them could we accommodate?

We mathematicians have the tools to work out a good solution.

Mathotopia: Some annoying real-world problems

- **Not even rational people always get along well with each other.**
  - We will let in as many immigrants as possible so that every two of them will be compatible and get along with each other.
- **Resources are always limited.**
  - We will leave it to our friends at the Economics Department to figure our how to feed, cloth, house and meaningfully employ all these immigrants.
- **It is not immediately obvious who will get along with whom.**
  - We will leave it to our friends at the Psychology Department to design a reliable questionnaire to determine who is compatible with whom. This will allow us to screen out some applicants and avoid all conflicts between the ones we accept.

Mathotopia's INS: A simple algorithm

We put **Dr. Sequential** in charge of Mathotopia's INS.
His office will use the following algorithm:

- Handle applications one-by-one.
- Let in the first applicant.
- After having accepted n applicants and possibly rejected some, screen the next application. If that applicant is compatible with all those that were already accepted, let her in; if not, reject the application.
- Iterate this process until all 10^9 applications are screened.

The politically sensitive issue of fairly screening applications from married couples and families is being handled by **Dr. Set.** Details of her procedure are classified and will not be discussed here.

Dr. Finite is our liaison to the Economics Department and will tell them about the expected number of successful applications.

What should Dr. Finite report to them?

Dr. Finite needs a mathematical model

Dr. Finite consults Harmony, a bright graduate student:

Dr. Finite: "Go over to the Psychology Department and figure out really quick what makes people get along with each other, so that we can put the info into a mathematical model and derive a ball-park figure for the economists."

Harmony: (Reports back.) "They told me it's complicated."

Dr. Finite: "Great! So we have a topic for your dissertation."

Harmony: "Thank you, Dr. Finite. But you told me that the report to Economics was due next week."

Dr. Finite: "Er . . . well . . . right!

We . . . we will figure out something.

I have to go to a faculty meeting right now.
Let's meet again tomorrow."

Dr. Finite and Harmony the next day

Dr. Finite: "There are extensive and detailed data on facebook unfriending . . ."

Harmony: "But they are propriatory."

Dr. Finite: "We could apply for a grant to buy access.
Helping me with the grant application will be a great professional development opportunity for you . . ."

Harmony: "But your report is due in six days."

Dr. Finite: "Right! Now I have a committee meeting . . ."

Harmony: "But wait! Since compatibility between humans is complicated, we could perhaps treat this relation as random and derive a quick estimate?"

Dr. Finite: "Er . . . of course! As your advisor, I need to give you some room for independent thinking instead of telling you all the right answers. I am glad you figured this one out by yourself."

Dr. Finite and Harmony keep talking

Dr. Finite: "Now you need to go to the Psychology Department again and see whether they have any relevant information about the probability distribution"

Harmony: "I already got this info. Estimates in the literature vary, but we can say confidently that each rational person is compatible with between 50 and 90 percent of all other rational persons."

Dr. Finite: "This is not enough to go by."

Harmony: "Could we assume independence of compatibility between different pairs of people?"

Dr. Finite: "This is almost certainly false, but the assumption is commonly made in ball-park estimates for complicated phenomena about which we have very little information."

Harmony: "Under this assumption, we can use the information to derive worst-case and best-case estimates."

Dr. Finite: "How?"

Harmony derives her estimates

Let p be the probability that a randomly chosen pair of rational people is compatible.

For each $n \geq 0$, assume we have already accepted n applicants, and let ξ_n be the number of applications that will be screened until we accept the next person.

Then $\xi_0 = 1$. For $n > 0$ we get by independence:

$P(\xi_n = k) = (1 - p^n)^{k-1} p^n$. Thus the expected value of ξ_n is $E(\xi_n) = \sum_{k=1}^{\infty} kP(\xi_n = k) = \sum_{k=1}^{\infty} k(1 - p^n)^{k-1} p^n = \frac{1}{p^n}$.

It follows that the expected number of applications that we would need to screen if we were to let in N applicants is

$\sum_{n=0}^{N-1} E(\xi_n) = \sum_{n=0}^{N-1} p^{-n} = \frac{p^{-N}-1}{p^{-1}-1} \approx \frac{p^{-N}}{p^{-1}-1}$.

We can set the right-hand side equal to 10^9 (the total number of expected applications), substitute the lower and upper bounds $p = 0.5$ and $p = 0.9$ for p that we inferred from the psychology literature, and solve for N.

Harmony and Dr. Finite reconnect with their friends

Harmony: "For $p = 0.5$ we get the lower estimate

$\frac{p^{-N}}{p^{-1}-1} = 2^N = 10^9$, so that $N = \log_2(10^9) \approx 30$.

For $p = 0.9$ we get the upper estimate

$\frac{p^{-N}}{p^{-1}-1} = 9\frac{10^N}{9^N}$, so that $N = \log_{10/9}(10^9) - \log_{10/9} 9 \approx 176$."

Dr. Finite: "Great! I am sure our colleagues at the Economics Department will be able to handle the resource issue.

Now you have the results for Chapter 1 of you dissertation. The remaining chapters will deal with the assumption of independence."

(Terminates the meeting with Harmony and pulls out his cell phone to tell his buddies that he will be able to play golf over the weekend after all.)

Harmony: (Pulls out her cell phone and excitedly texts the news to her friend Equality.)

Ten seconds later

(Ringtone composed of the first 40 digits of π.)

Equality: "Glad about your first chapter. But this is horrible!!!"

Harmony: (Startled.) "What??"

Equality: "Mathotopia is all about giving those billion rational people a chance to live in peace and dignity! And you found that we will accept only about 100 of them!!
Only 1 tenth of 1 percent of 1 percent of 1 percent of them!!!"

Harmony: (With a lump in her throat.) "But this is what the model predicts. I only did the calculations."

Equality: (Controls her emotions.) "Look, I don't blame you. Or your model. In fact, now that I think it through, you have done all of us a favor by pointing out that there is something wrong with how we want to populate mathotopia."

Harmony: (Regains her composure.) "Do you mean there could be a better procedure for screening the applications?"

Equality and Harmony call Sol Optimal

Equality: "Perhaps there is.
Let's do a three-way skype with Sol Optimal and ask him."

Harmony: "I don't like this guy."

Equality: "Me neither. But he knows how to make things better."

Sol Optimal: "Hey girls! What's up?"

Equality: "What do you think about the screening procedure for applicants to mathotopia?"

Sol Optimal: "It sucks. This guy Sequential is a dirty old ..."

Harmony: "Sol!! Cut it out!"

Equality: "Can you tell us what, exactly, is wrong with the procedure?"

Sol Optimal: "Glad somebody finally asks me this question.
The algorithm doesn't necessarily find the optimal solution."

Harmony and Equality: (Sweetly.) "So now that we have come to the right person, can you explain this to us?"

Sol Optimal explains

Sol Optimal: "Let's represent the applicants as nodes of a graph. Draw an edge between any two given nodes if the applicants represented by these nodes are compatible.

We want to populate mathotopia with applicants who form a clique in this graph, that is, a subset of nodes such that each two of them are connected with an edge."

Equality: "And we want this set to be as large as possible."

Sol Optimal: "Exactly. We want to solve the combinatorial optimization problem of finding a clique of maximal size. Mr. Sequential's algorithm will always find a clique, but not necessarily one of maximal size."

Harmony: "I don't like calling the people of mathotopia a clique."

Sol Optimal: "Then draw an edge when two applicants are incompatible and look for an independent subset of the graph of maximal size."

Equality: "This is really the same mathematical problem."

How about implementation?

Harmony: "I can see now that Dr. Sequential's algorithm is not the best."

Sol Optimal: "I would call it quick and dirty.

It runs fast, but is likely to miss the optimal solution."

Equality: "So what would be the optimal algorithm for finding a largest clique,"

Harmony: (Frowns.)

Equality: ". . . or, if you prefer, largest independent set?"

Sol Optimal: (Laughs.) "Ask Dr. Complexity over at CS.
Gotta go now. Been a pleasure talking with you."
(Quickly signs off.)

Harmony: (Sigh of relief.)

Equality: "As I said, I don't like him either.

But let's give Dr. Complexity a ring."

The conversation with Dr. Complexity

Harmony: "Good morning, Dr. Complexity! Could you spare a few minutes to talk with us?" (Fills her in on the details.)

Dr. Complexity: "Sol Optimal was right."

Equality: "So how could we find the optimal solution?"

Dr. Complexity: "You could in principle test all subsets of nodes one by one and determine which ones form cliques.
Then output the one with the largest size."

Harmony: "But we would need to look at 2^{10^9} sets of nodes!!
This seems not feasible even on the largest supercomputer."

Dr. Complexity: "Exactly."

Equality: "Is there an efficient way of solving this problem?"

Dr. Complexity: "Most likely not. The problem of finding the maximal clique in a graph is one of many so-called NP-hard problems, and all the available evidence indicates that it does not have a computationally feasible solution."

What can we do?

Equality: (Dejected.) "Do you mean that Dr. Sequential's algorithm is the only feasible one?"

Dr. Complexity: "No. There are better and still feasible algorithms. Finding improved approximation algorithms for NP-hard optimization problems is an active area of research."

Harmony: "But algorithms that find better solutions will, in general, be more expensive to run?"

Dr. Complexity: "Right! Need to go now.
Good luck with your project!"

Harmony and Equality: "Thank you, Dr. Complexity!"

Equality: "I would throw all computational resources of the world at the problem of making as many people as possible happy in mathotopia!!!"

Harmony: "But how much better would the optimal solution likely be? Could we accept thousands rather than a hundred people?
I guess I need to work some more on Chapter 1 of my dissertation."

Harmony's new Chapter 1

Let p be the probability that a randomly chosen pair of rational people is compatible, and let N be the total number of applicants.

For each $n < N$ and set S of size n of applicants, let ξ_S be the random variable that takes the value 1 if all applicants in S are compatible, and takes the value 0 otherwise.

Then $E(\xi_S) = P(\xi_S = 1) = p^{n(n-1)/2}$.

Now let η_n be the sum of all ξ_S for S of size n. Then

$E(\eta_n) = \sum_S \xi_S = \binom{N}{n} p^{n(n-1)/2} < N^n p^{n(n-1)/2} = \left(Np^{(n-1)/2}\right)^n$.

Harmony realizes that a person like Sol Optimal would call her estimate of $\binom{N}{n}$ in the above derivation "quick and dirty." She will later on improve it using Stirling's formula in her clean write-up for Dr. Finite, but for now she just wants a ball-park estimate.

She notices that $E(\eta_n) > P(\eta_n \geq 1)$, so that $\left(Np^{(n-1)/2}\right)^n$ is an upper bound for the probability that there is any clique of size $\geq n$ in the graph that represents the problem.

Harmony's ballpark estimate for Chapter 1

Let us consider $\left(Np^{(n-1)/2}\right)^n \approx \left(Np^{n/2}\right)^n$,

where the approximation might be reasonably good for large n.

When N is significantly less than $p^{-n/2}$, then this probability will be very, very small.

For $p = 0.5$, by symmetry, we can conclude that the vast majority of graphs with $< \sqrt{2^n}$ nodes will contain **neither** a clique **nor** an independent set of size $\geq n$.

For $N = 10^9$ and $p = 0.5$, it is very unlikely that mathotopia could accept $n = 2\log_2(10^9) \approx 60$ or more applicants;
for $N = 10^9$ and $p = 0.9$, it is very unlikely that mathotopia could accept $n = 2\log_{10/9}(10^9) \approx 393$ or more applicants.

An optimal screening algorithm might increase the number of successful applicants by approximately a factor of 2, but not by an order of magnitude.

Stunned by her findings ...

Harmony stares at her result in disbelief.

An invisible force draws her hand to the phone,

but she first double-checks her calculations.

They are correct.

Only one fifth of 1 percent of 1 percent of 1 percent!!!

She reaches again for the phone, but can't bring herself to dial

Equality and breach the news to her.

Harmony: (Deep sigh.)

She decides to make the best of a bad situation and writes up the results.

Dr. Finite and Harmony meet on Monday morning

(As usual, Dr. Finite's golfing performance over the weekend was below par of what he believes to be his potential.)

Dr. Finite: (Grouchy.) "Anything new?"

Harmony: "I extended the results for Chapter 1."
(Shows her write-up).

Dr. Finite: "Hmm. OK."

Harmony: (Timidly.) "Could this be the final version of Chapter 1?"

Dr. Finite: "Well . . . as far as I am concerned . . .

But there is this guy Dr. Ramsey on your committee.
He can be a bit . . . you know.

Better run this by him first for approval."

Harmony: "Dr. Ramsey, would you be so kind as to take a look at Chapter 1 of my dissertation? Dr. Finite thinks it is ok."

Dr. Ramsey: (Quickly leafs through the write-up.) "Well, my dear Miss Harmony, the knowledge of the mathematical literature of my esteemed colleague Dr. Finite is rather, shall we say, ... limited."

Harmony: "Wouldn't that be true for each of us?"

Dr. Ramsey: "Well spoken, young lady! Very well spoken!

What you have rediscovered here is an old result of Paul Erdős. It is related to an even older theorem of mine.

Highly remarkable, young lady! Very remarkable indeed!"

Harmony: (Taken aback.) "Can you explain the theorem to me?"

Dr. Ramsey explains his theorem

Dr. Ramsey: "With great pleasure. The theorem is more general, but in the special case that is of interest here, it says that for every positive integer n, here exists an integer $R(n)$ such that every graph with $R(n)$ nodes either contains a clique of size n or an independent set of size n."

Harmony: "But this doesn't define $R(n)$. Wouldn't every larger integer have the same property?"

Dr. Ramsey: "Very acutely observed, young lady!
We use $R(n)$ for denoting the smallest such number."

Harmony: "Is $R(n)$ called a Ramsey number?"

Dr. Ramsey: "With all due modesty, yes indeed! There are other types of Ramsey numbers for the more general case of my theorem, but we will focus here on the ones defined above."

Dr. Ramsey: "When we consider how fast $R(n)$ grows, then the result that you rediscovered ..."

Harmony: "shows that $\liminf_{n\to\infty} (R(n))^{1/n} \geq \sqrt{2}$.

But this is not publishable because it was already known."

Dr. Ramsey: (Nods quietly.)

Harmony: (A tad irritated.) "So why did you call my result remarkable?"

Dr. Ramsey: "Because, my dear Miss Harmony, Paul Erdős, who first discovered this proof, was one of the greatest mathematicians of all times.

It is no mean achievement to rediscover one of his theorems."

Harmony: "But this proof is weird! It does not give us any actual large graph without cliques and independent sets of size n, it only shows that such graphs exist!"

An open problem

Dr. Ramsey: "Very keenly observed, young lady! What you would like to have is called a constructive proof of the same result."

Harmony: "Where can I find such a constructive proof?"

Dr. Ramsey: "In your own head. None exists in the literature."

Harmony: "So, if I find a constructive proof, this would in your opinion suffice for Chapter 1 of my thesis?"

Dr. Ramsey: (Smiles.) "And you could skip Chapters 2 and higher."

Harmony: "What do you mean?"

Dr. Ramsey: "Finding a constructive proof that

$\liminf_{n\to\infty} (R(n))^{1/n} > 1$

is a famous open problem.

Paul Erdős offered a prize of $100 for such a proof."

Erdős problems

Harmony: "Are you joking?"

Dr. Ramsey: "Not at all. Paul Erdős posed many open problems, often with small financial rewards.
Very few have ever been solved.
Solving any one of them would be sufficient for a Ph.D. thesis at any university."

Harmony: "Wow!"

Dr. Ramsey: "For example, the best known asymptotic upper bound for my numbers is
$\limsup_{n\to\infty} (R(n))^{1/n} \leq 4$.

In 1947 Erdős offered \$100 for showing that
$\lim_{n\to\infty} (R(n))^{1/n}$ exists,
and \$250 for finding its value if it does."

Erdős problems and mere mortals

Harmony: "Where can I read more about Erdős problems?"

Dr. Ramsey: "At
http://www.math.ucsd.edu/~erdosproblems/All.html
Any one of these problems will be a good one for you to work on."

Harmony: "With all due respect, Dr. Ramsey, this is not fair. You said that very few of these problems have ever been solved. I am only a beginning graduate student at OU! These problems are for accomplished mathematicians at Harvard, Berkeley, or Cambridge!"

Dr. Ramsey: "Well, there is at least one known case of a \$100 Erdős problem that was solved by a Master's student at an Eastern European university, not even at Oxford, let alone Cambridge … In any case, the point is to gain deep insights by trying."

Harmony: (Feeling a bit dizzy.) "Thank you, Dr. Ramsey, for your valuable feedback!"

(Leaves, and with her head still spinning, picks up the phone.)

Harmony calls Equality and confides everything

Harmony: (Breaks the bad news about mathotopia.)

Equality: “So fixing the screening algorithm won't help much?”

Harmony: “Unfortunately, no.”

(Deep and profound silence.)

Harmony: (Tells about the Erdős problems.)

Equality: (Getting angry.) “Are you going to work on this stuff?”

Harmony: “Yes!!”

Equality: (Losing it.) “Why??? That white male Erdős is undercutting minimum wage!!! And you may ruin your career!!!

Harmony: Because these problems interest me. Forget the $$.

Equality: **?????** ???? Oh?! !!! That's it! **!!!!!!!!**

What happened here to Equality?

Seven years later: Our administrators

- Dr. Sequential took early retirement.

  He devotes all his energy to practicing serial monography.

- Dr. Set now works for the NSA.

  No specific details are known.

- Dr. Finite is now Provost at Hol(e)y Lawn College, an institution consistently ranked among the top five in the nation for its on-campus golfing and clubbing facilities.

  His performance at golf is still below par of what he believes to be his potential.

  With his college being deep in the hole, the same might be said about his performance as Provost.
  But he never noticed.

Seven years later: Our faculty

- Dr. Complexity is believed to have solved the problem of how to simultaneously be Department Chair, carry out a successful research program, excel in teaching, and keep three teenage sons out of trouble.

  Alas, we were unable to schedule an interview with her.

- Sadly, Dr. Frank P. Ramsey passed away in 1930, at the age of 26 in Cambridge, UK.

  His theorem lives on.

  It continues to inspire generation after generation of students.

Seven years later: Our students

- Dr. Harmony wrote a brilliant dissertation.

  She has not solved any problem of Erdős yet, but keeps trying.

- Dr.h.c. Equality left mathematics and switched to sociology and political science.

  She led a successful campaign to change the compatibility-based criteria for citizenship of mathotopia to one based on genuine interest in and enthusiasm for mathematics or a related field.

  This has saved many, many lives from the ravages of irrationality.

  Join the millions of her grateful followers on Twitter:

  # Solve-the-hardest-open-problem: disagree-and-getalong

Seven years later: Sol Optimal

- Sol Optimal never bothered to obtain a formal degree, seek employment, or pay income tax.

  He keeps himself afloat by accepting donations for consulting services from people with unsolvable time-management problems.

  There are unconfirmed rumors that Dr. Complexity is among his satisfied sponsors.

Seven years later: Mathotopia

Mathotopia prospers, with millions of citizens who lead productive and satisfying lives.

Dr.h.c. Equality is generally revered as the Founding Mother.

Both Dr. Harmony and Sol Optimal are proud citizens.

They still don't like each other, but get along pretty well based on grudging (mind you!) mutual respect.

Neither of them would rule out the possibility of one day writing a joint paper.

Any suggestions for a topic of that paper?

The student who solved an Erdős problem

The student of Warsaw University who solved an Erdős problem in 1980 recently got invited to give a talk in mathotopia.

It was widely expected that he would give a presentation of profound mathematical depth.

Instead, he staged a silly little play.