Discrete Dynamical Systems: The Linear, the Nonlinear, and the Chaotic Part II

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A (determinisitic) **discrete-time dynamical system** is a pair (X, F) such that

- The state space X is a topological space.
- $F : X \to X$ is a continuous map.

The time evolution function is then given by

$\varphi(x, t) = F^t(x)$.

When F is a homeomorphism we will write (X, T) instead of (X, F). The system then becomes time-reversible and we take $\mathbb{T} = \mathbb{Z}$ as the time line.

When we write (X, F), we implicitly assume that $\mathbb{T} = \mathbb{N}$ is the time line.

Review: Linear systems

Consider systems $(\mathbb{R}^n, F)$. For simplicity we will mostly write x instead of $\vec{x}$.

- Such a system is **linear** if $F(x) = Mx$ for some $n \times n$ matrix M.
- The zero vector $\vec{0}$ is always an equilibrium, and it is unique iff $M - I$ is invertible, that is, iff 1 is not an eigenvalue of M.
- Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of M.
  - $\vec{0}$ is (locally and globally) asymptotically stable iff $\max |\lambda_i| < 1$,
  - $\vec{0}$ is unstable if $\max |\lambda_i| > 1$,
  - If all eigenvalues have multiplicity 1, then $\vec{0}$ is Lyapunov stable iff $\max |\lambda_i| \leq 1$.

In the previous lecture we looked at an example of dimension $n = 1$; now we will look at a higher-dimensional example.

Example 5: Age-structured populations

The simplest model of population growth under unlimited resources assumes that $P(t+1) = \lambda P(t)$, where $P(t)$ is the population at time t and λ is a positive constant.

This ignores the fact that not all age groups contribute equally to population growth. In age-structured models, the (female) population is partitioned into age classes $\vec{P} = (P_1, \dots, P_n)$.

For each i, let σ_i be the survival probability for individuals in the i-th class for one time step. Moreover, let β_i be the average number of daughters that an individual in the i-th class contributes to P_1 over one time unit. This gives:

$$P_1(t+1) = \sum_{i=1}^{n} \beta_i P_i(t)$$

$$P_{i+1}(t+1) = \sigma_i P_i(t) \quad \text{for } i < n$$

$$P_n(t+1) = \sigma_{n-1} P_{n-1}(t) + \sigma_n P_n(t).$$

Example 5 continued: The Leslie matrix

The dynamics defined on the previous slide can be written as $\vec{P}(t+1) = L\vec{P}(t)$, where L is called the **Leslie matrix.**

Fore example, consider a population of human females (in millions) with age classes 0–19, 20–39, 40–59, 60–120. If the mortalities over a 20-year time step for these age classes are 10%, 20%, 30%, 70% respectively, and the average numbers of daughters are $\beta_1 = 1, \beta_2 = 0.5, \beta_3 = 0.02, \beta_4 = 0$, then we get the Leslie matrix

$$L = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

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$$L = \begin{bmatrix} 1 & 0.5 & 0.02 & 0 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ ? & ? & ? & ? \end{bmatrix}$$

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$$L = \begin{bmatrix} 1 & 0.5 & 0.02 & 0 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 \end{bmatrix}$$

Example 5 continued: Properties of L

$$\vec{P}(t+1) = L\vec{P}(t), \text{ where } L = \begin{bmatrix} 1 & 0.5 & 0.02 & 0 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 \end{bmatrix}$$

- $\vec{v}_1 = (0,0,0,1)^T$ is an eigenvector with eigenvalue 0.3.
- $\vec{v}_2 = (0.4279, 0.2867, 0.1708, 0.1146)^T$ is an eigenvector with eigenvalue 1.3430 and $\|\vec{v}_2\|_1 = 1$.
- There are two more eigenvectors $\vec{v}_3, \vec{v}_4$ with eigenvalues -0.3083 and -0.0348 respectively.
- The **stable subspace** $E_s = span(\vec{v}_1, \vec{v}_3, \vec{v}_4)$.
- The **unstable subspace** $E_u = span(\vec{v}_2)$.
- The equilibrium $\vec{0}$ is **hyperbolic.**

Example 5 continued: Properties of the dynamics

$$\vec{P}(t+1) = L\vec{P}(t), \text{ where } L = \begin{bmatrix} 1 & 0.5 & 0.02 & 0 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 \end{bmatrix}$$

- The biologically feasible region Ω consists of all vectors that have only nonnegative coordinates.
- The intersection of this region with E_s is contained in $span((0,0,0,1)^T)$. All trajectories that start in this area asymptotically approach $\vec{0}$.
- For all trajectories that start in $\Omega \setminus E_s$ we have $\lim_{t\to\infty} P_i(t) = \infty$.
- Moreover, for all trajectories that start in $\Omega \setminus E_s$ we have $\lim_{t\to\infty} \frac{\vec{P}(t)}{\|\vec{P}(t)\|_1} = \vec{v}_2$, where $\|\vec{P}(t)\|_1 = |P_1(t) + P_2(t) + P_3(t) + P_4(t)|$.

Example 6: A nonlinear version of Example 5

Let L be the Leslie matrix of Example 5, and let

$\Omega_1 = \{\vec{P} \in \Omega : \; P_1 + P_2 + P_3 + P_4 = 1\} = \{\vec{P} \in \Omega : \; \|\vec{P}\|_1 = 1\}.$

Consider the system (Ω_1, T), where $T(\vec{P}) = \frac{L\vec{P}}{\|\vec{P}\|_1}$.

During the presentation we numerically explored this system.

$\vec{P}^* = (0, 0, 0, 1)^T$ is an unstable equilibrium.

$\vec{P}^{**} = (0.4279, 0.2867, 0.1708, 0.1146)^T$ is a locally stable equilibrium.

It is approached by all trajectories that start with $\vec{P}(0) \neq \vec{P}^*$.

While the system (Ω_1, T) is nonlinear, it has the big advantage that its state space Ω_1 is a compact subset of $\Omega \subset \mathbb{R}^4$.

Most of the theory of dynamical systems focuses on systems with a compact state space.

Example 7: The discrete logistic system

All population dynamics models that we have considered so far implicitly assume unlimited resources. The **discrete logistic model** is based on the more realistic assumption that at high population densities the population will grow at a slower rate or even decline due to scarcity of resources.

The variable x in this model represents the population size as a fraction of a hypothetical maximal population size, so that the state space $X = [0, 1]$ is compact.

The updating function is given by $F(x) = rx(1 - x)$, where r is a parameter with $0 < r \leq 4$.

When $x_t \approx 0$ we have $x_{t+1} = rx_t(1 - x_t) \approx rx_t$, so that population growth will be nearly exponential.

At higher population densities the factor $1 - x_t$ will slow down growth, and when $x_t > 0.5$, result in a population decline between times t and $t + 1$.

Properties of the discrete logistic system

The properties of the dynamics depend on the value of r:

- For $0 < r \leq 1$, the system has a unique equilibrium $x^* = 0$ (extinction). It is asymptotically stable.
- For $1 < r$, the equilibrium $x^* = 0$ is unstable and the system has a second equilibrium x^{**} that corresponds the **carrying capacity of the environment.**
- For $1 < r < 3$ the equilibrium x^{**} is locally asymptotically stable and will be approached by all trajectories that start in $(0, 1]$.
- For $3 < r$ the equilibrium x^{**} is unstable.
- For $3 < r < 1 + \sqrt{6} \approx 3.4495$ the system has an orbit of period 2 that will be approached by almost all trajectories.

We can see that the system undergoes qualitative changes called **bifurcations** when we increase the **bifurcation parameter** r past certain **bifurcation values** $r^* = 1, 3, 3.4495$.

The one at $r^* = 3.4495$ is called a **period-doubling bifurcation.**

The discrete logistic system for larger values of r

When we slowly increase the value of r beyond 3.4495:

- First a sequence of additional period-doubling bifurcations occurs when we increase r on $(3.4495, 3.570)$,
  so that successively stable periodic orbits of length 2^n appear and then become unstable .
- For $r \in (3.570, 4]$ the situation is quite complicated.
  - For example, when $r \approx 3.839$, there is a unique stable equilibrium of period 3.
  - For some other values of r in this range, the system exhibits **chaotic dynamics.**

But what, exactly, is chaos?

First hallmark of chaos: Sensitive dependence

The literature on definitions of chaos is a bit ... chaotic.

There are many different, not necessarily equivalent definitions.

One important feature is **sensitive dependence,** which means that trajectories that start nearby will quickly grow far apart.

Here is one formal definition.

Definition

Let (X, F) be a discrete dynamical system and let $K \subseteq X$. We say that the system is **sensitive on** K if

$$\exists \varepsilon > 0 \forall x \in K \, \forall \delta > 0 \exists y \in B_\delta(x) \cap K \, \exists t \in \mathbb{N} \; d(F^t(x), F^t(y)) > \varepsilon.$$

Sensitive dependence alone does not characterize chaos

A linear system is sensitive (on the whole state space) iff the equilibrium $\vec{0}$ is unstable.

These examples would not be considered chaotic though, as trajectories that grow apart will keep growing apart, at an exponential rate, in a very predictable way.

If the state space is compact, then there is an upper limit on how far apart two trajectories can grow.

In truly chaotic systems, trajectories may grow apart for a while, but then get close together again in a region K of the state space where we have sensitive dependence, and this pattern will repeat *ad infinitum.*

Transitivity

Definition

(X, F) is **transitive** if for all nonempty open $U, V \subseteq X$ there exist infinitely many $t > 0$ with $F^t(U) \cap V \neq \emptyset$.

- Linear systems are not transitive.
- The discrete logistic system $([0,1], F)$ with $F(x) = 4x(1-x)$ is transitive and sensitive on $K = [0,1]$.

Definition (First definition of chaos)

A system (X, F) that is transitive and sensitive on X is chaotic.

A second definition of chaos

Our first definition is too restrictive, as it does not apply to those examples of systems where trajectories approach a subset of the state space called an **attractor.** These attractors are often, but not always, beautiful **fractals.**

These attractors may be proper subsets $A \subset X$, and forward trajectories will eventually move out of any open set U whose closure does not intersect A, so that we don't have transitivity on the whole state space.

But attractors will always be forward invariant, so that $(A, F \restriction A)$ is a related dynamical system.

Definition (Second definition of chaos)

A system (X, F) is chaotic iff it is sensitive on an attractor A.

What are attractors? Examples and an informal definition

- For example, if x^* is a locally asymptotically stable fixed point, then $\{x^*\}$ is an attractor.
- More generally, if x is locally asymptotically stable fixed point of (X, F^p), then $\{x, F(x), F^2(x), F^{p-1}(x)\}$ is a periodic attractor.
- Attractors are closed and forward invariant subsets of X.
- A attracts a nonempty open set $U \supseteq A$ of initial conditions. For any trajectory that starts in U we have $\lim_{t\to\infty} d(x(t), A) = 0$.
- A is minimal with respect to the last two properties.

Assume a dynamical system is sensitive on an attractor.

- The dynamics on the attractor cannot be periodic. In particular, the attractor cannot be a finite set.
- Aperiodicity on the attractor does not all by itself imply chaos. For example, consider a rotation of $X = S^1$ by an angle α such that $\frac{\alpha}{2\pi}$ is irrational. Such a system is transitive, has aperiodic and dense orbits, but is not sensitive.
  Here S^1 itself is the only attractor.
- If (X, F) is transitive, then X itself must be the attractor. While many attractors of chaotic systems are fractals, some are perfectly ordinary sets, like $[0, 1]$ in the discrete logistic system $([0, 1]), 4x(1 - x))$.