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PARTIAL DECOUPLING OF THE MATRIX METHOD FOR PLANAR MECHANISMS INVERSE DYNAMICS

Robert L. Williams II, Ph.D.
Professor

Department of Mechanical Engineering, Ohio University
williar4@ohio.edu

ABSTRACT

This paper presents a partial decoupling method for the well-known matrix method commonly in use for the linear inverse dynamics problem of planar mechanisms. Any mechanism with a dyad of binary links can benefit from the proposed method, which involves extracting submatrices by looking for sufficient columns of zeros such that a reduced problem may first be solved with an equal number of scalar equations and unknowns. Then some equal-and-opposite internal joint forces are transferred to the remaining FBDs and the solution proceeds until the input link is solved in a decoupled manner. The method leads to significant reductions in computational cost for common planar mechanisms. The kinematics & dynamics textbooks have overlooked this partial decoupling in their presentations of the matrix method for inverse dynamics.

1. INTRODUCTION

The inverse dynamics problem (also referred to as kinetostatic analysis) for planar mechanisms is stated as follows: Given the mechanism, desired kinematic motion, and external loads to overcome, find the unknown internal joint forces and the driving torque (or force for a translating input motor). The solution of this problem is an important step in the machine design process, to size links to withstand the dynamic loading and to size the actuator. With the kinematic terms given, the dynamic equations of motion are linear in the unknown forces and driving torque.

The old-school solution technique was the method of superposition (e.g. [1]). The inverse dynamics problem was solved link-by-link, where only one moving link at a time was given mass and rotational inertia. The remaining links became two-force members and the problem was solved for each link graphically or analytically. When all links had been assigned inertia in their turn, the overall solution was found by summing the unknowns for all steps. This method was nice due to the physical insights gained; however, it took a long time to solve a single inverse dynamics snapshot.

In the 1980s the matrix method gained prominence to solve the inverse dynamics problem [1,4,5]. This is a general procedure to assemble all the dynamics equations of motion into a single matrix-vector equation (three scalar equations for each moving link, XY sum of forces and Z moment), linear with n equations in n unknowns. Dynamicists were taking advantage of the power of computers (as PCs were on the rise) to solve linear equations via standard matrix methods. Physical insight and analytical solutions were sacrificed, but the computer could solve the inverse dynamics problem quickly and in one step at each snapshot. Using the computer, the inverse dynamics problem was easily solved for the entire range of motion of any planar mechanism, to find the worst cases for mechanical design.

In teaching about 900 students to date, the matrix method has been the author's choice for solving the inverse dynamics problem. Recently we made a discovery regarding a partial decoupling in the matrix method for planar mechanisms, not presented in any of the textbooks on the subject [1-7].

2. FOUR-BAR MECHANISM INVERSE DYNAMICS MATRIX METHOD

For each moving link we can write the following two vector dynamics equations (Newton's 2nd Law and Euler's Equation):

$$\begin{aligned} \sum \mathbf{F}_i &= m_i \mathbf{A}_{G_i} \\ \sum \mathbf{M}_{G_i} &= I_{G_i} \boldsymbol{\alpha}_i \end{aligned} \quad (1)$$

Vector equations (1) yield two XY scalar equations from the dynamic force balance and one Z scalar equation from the dynamic moment balance. Moments must be summed about a suitable point (chosen as the center of gravity for each link above; the mass moment of inertia must be for the same point).

For the standard four-bar mechanism, three free-body diagrams (FBDs) are drawn and equations (1) are applied, once for each moving link. The result is a 9x9 coupled set of linear equations (2) to solve for the unknowns (four vector internal joint forces \mathbf{F}_{ij} and the driving torque τ_2). A similar set of equations is found in [1-4]. Again, there are three scalar equations for each moving link, for a total of 9 equations. There are eight scalar force unknowns (4 vector internal forces with 2 XY components each) and one scalar torque unknown for a total of 9 scalar unknowns.

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ r_{12Y} & -r_{12X} & -r_{32Y} & r_{32X} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & r_{34Y} & -r_{34X} & -r_{14Y} & r_{14X} & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14X} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G2X} \\ m_2 (A_{G2Y} + g) \\ I_{G2Z} \alpha_2 \\ m_3 A_{G3X} - F_{E3X} \\ m_3 (A_{G3Y} + g) - F_{E3Y} \\ I_{G3Z} \alpha_3 - r_{E3X} F_{E3Y} + r_{E3Y} F_{E3X} - M_{E3} \\ m_4 A_{G4X} - F_{E4X} \\ m_4 (A_{G4Y} + g) - F_{E4Y} \\ I_{G4Z} \alpha_4 - r_{E4X} F_{E4Y} + r_{E4Y} F_{E4X} - M_{E4} \end{Bmatrix} \quad (2)$$

In (2), \mathbf{F}_{ij} is the unknown vector internal joint force of link i acting on link j , \mathbf{r}_{ij} is the known moment arm vector pointing to the R joint connection with link i from the CG of link j , gravity g is included, and external vector forces \mathbf{F}_E and moments \mathbf{M}_E are included for links 3 and 4.

This 9x9 set of linear equations can be represented by $\mathbf{A}\mathbf{v} = \mathbf{b}$ and the solution is, conceptually, $\mathbf{v} = \mathbf{A}^{-1}\mathbf{b}$. Gauss-Jordan elimination with pivoting is much more efficient and

numerically robust and it should be used in place of matrix inversion.

3. PARTIAL DECOUPLING OF THE MATRIX METHOD

3.1 DR. BOB PEDAGOGY

Prior to approaching a useful mechanism like the four-bar (Figure 1) in class, I always introduce the inverse dynamics solution of a single rotating link. This involves a 3x3 matrix-vector equation, an easier step for the students to grasp on the way to the 9x9 four-bar dynamic equations (2). The problem may be solved without matrix methods, as the X and Y forces are decoupled.

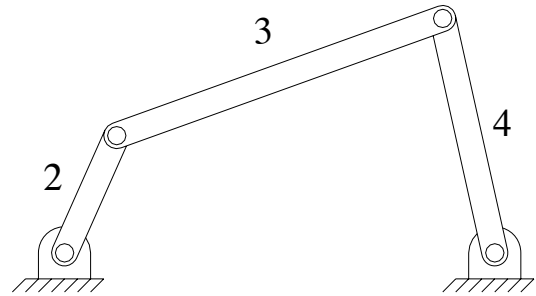


Figure 1. Standard Four-Bar Mechanism

In leading up to the derivation of (2) for the four-bar mechanism inverse dynamics problem, I always ask the class if we can decouple this problem link-by-link and thus make use of the simpler single rotating link dynamics. Link 2 has 3 equations and 5 unknowns; links 3 and 4 have 3 equations and 4 unknowns each. So until recently, the answer was always no, we cannot solve one link at a time, the links are coupled. This statement is still true. But then an ME junior asked, can't we look at the link 3 and 4 FBDs together? This would be 6 equations in 6 unknowns.

3.2 FOUR-BAR MECHANISM DECOUPLING

Looking at the original 9x9 matrix (2) and noting there are three 6x1 columns of zeros (columns 1, 2, 9) in rows 4 through 9, it is clear that links 3 and 4 may be solved together, without link 2. Here is a more efficient solution, compared to the full 9x9 system of equations (2):

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & r_{34Y} & -r_{34X} & -r_{14Y} & r_{14X} \end{bmatrix} \begin{Bmatrix} F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14X} \\ F_{14Y} \end{Bmatrix} =$$

$$\begin{Bmatrix} m_3 A_{G3X} - F_{E3x} \\ m_3 (A_{G3Y} + g) - F_{E3y} \\ I_{G3Z} \alpha_3 - r_{E3x} F_{E3y} + r_{E3y} F_{E3x} - M_{E3} \\ m_4 A_{G4X} - F_{E4x} \\ m_4 (A_{G4Y} + g) - F_{E4y} \\ I_{G4Z} \alpha_4 - r_{E4x} F_{E4y} + r_{E4y} F_{E4x} - M_{E4} \end{Bmatrix} \quad (3)$$

(3) cannot be further decoupled since both sets of two 3x1 columns of zeros are insufficient.

Denote (3) as $\mathbf{A}_{34} \mathbf{v}_{34} = \mathbf{b}_{34}$ and solve for the six unknowns $\mathbf{v}_{34} = \mathbf{A}_{34}^{-1} \mathbf{b}_{34}$ (use Gauss-Jordan elimination in place of inversion) and then use the solved F_{32X} and F_{32Y} in the following 3x3 set of linear equations, from the link 2 FBD, similar to the single rotating link:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ r_{12Y} & -r_{12X} & 1 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G2X} - F_{32X} \\ m_2 (A_{G2Y} + g) - F_{32Y} \\ I_{G2Z} \alpha_2 + r_{32Y} F_{32X} - r_{32X} F_{32Y} \end{Bmatrix} \quad (4)$$

We do not need a matrix solution to (4) since the X and Y force equations are decoupled:

$$\begin{aligned} F_{21X} &= -m_2 A_{G2X} + F_{32X} \\ F_{21Y} &= -m_2 (A_{G2Y} + g) + F_{32Y} \\ \tau_2 &= I_{G2Z} \alpha_2 + r_{32Y} F_{32X} - r_{32X} F_{32Y} - r_{12Y} F_{21X} + r_{12X} F_{21Y} \end{aligned} \quad (5)$$

Matrix inversion requires approximately $\frac{3n^3}{\log n}$ and Gauss-Jordan elimination requires approximately $\frac{(n^2 - 1)n}{3} + n^2$ multiplications/divisions [8]. Table I presents the computational efficiency for solving the four-bar mechanism inverse dynamics in four ways.

Table I. Number of Multiplications/Divisions (Four-Bar)

Method	Inversion	Gauss-Jordan	Reduction
9x9	2292	321	86%
6x6 & link 2	840	113	87%
Reduction	63%	65%	

There is an impressive **65% reduction in computational cost** for Gauss-Jordan elimination with the 6x6 plus decoupled link 2 method compared to the full 9x9 approach in common use.

Also, the numerical accuracy may also improve with this method since we needn't do unnecessary calculations with the three 6x1 columns of zeros.

Clearly, Gauss-Jordan elimination requires far less computational power than matrix inversion. In addition, with the use of pivoting, Gauss-Jordan elimination leads to better numerical robustness. The CS computational efficiency experts at Ohio University say one can use the number of multiplications/divisions as a comparison basis; additions/subtractions are about as expensive and overall, they would roughly double the number of operations in Table I for each category, leading to the same reductions.

3.3 SLIDER-CRANK MECHANISM DECOUPLING

There is a similar partial decoupling for the slider-crank mechanism (Figure 2). The slider-crank mechanism inverse dynamics matrix-vector set of equations (6) is of dimension 8x8. Compared to the four-bar, the link 4 moment balance dynamic equation has been lost (it is $0 = 0$ due to zero moment arms and zero angular acceleration α_4). To address this problem, unknown F_{14X} was eliminated using a friction constraint at the piston wall [3]. There is no vertical translation of the slider ($A_{G4Y} = 0$ in (6)).

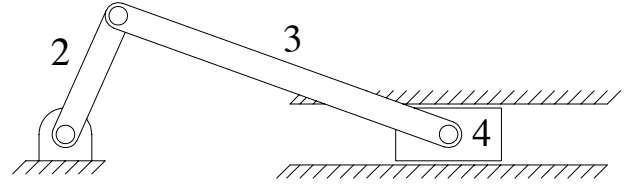


Figure 2. Standard Slider-Crank Mechanism

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ r_{12Y} & -r_{12X} & -r_{32Y} & r_{32X} & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \pm\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} =$$

$$\begin{Bmatrix} m_2 A_{G2X} \\ m_2 (A_{G2Y} + g) \\ I_{G2Z} \alpha_2 \\ m_3 A_{G3X} - F_{E3x} \\ m_3 (A_{G3Y} + g) - F_{E3y} \\ I_{G3Z} \alpha_3 - r_{E3x} F_{E3y} + r_{E3y} F_{E3x} - M_{E3} \\ m_4 A_{G4X} - F_{E4x} \\ m_4 g - F_{E4y} \end{Bmatrix} \quad (6)$$

In (6) $\pm\mu$ indicates the program must reverse the sign of the slider velocity on the friction coefficient μ for each snapshot in time. In the original 8x8 matrix (6) there are three 5x1 columns of zeros (columns 1, 2, 8) in rows 4 through 8.

So we may extract a 5x5 matrix-vector equation, solving for the link 3 and 4 FBDs independently of link2. Here is a more efficient solution:

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 \\ 0 & 0 & -1 & 0 & \pm\mu \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \end{Bmatrix} = \quad (7)$$

$$\begin{Bmatrix} m_3 A_{G3X} - F_{E3x} \\ m_3 (A_{G3Y} + g) - F_{E3y} \\ I_{G3Z} \alpha_3 - r_{E3x} F_{E3y} + r_{E3y} F_{E3x} - M_{E3} \\ m_4 A_{G4X} - F_{E4x} \\ m_4 g - F_{E4y} \end{Bmatrix}$$

Denote (7) as $A_{34} \mathbf{v}_{34} = \mathbf{b}_{34}$, solve for the six unknowns \mathbf{v}_{34} using Gauss-Jordan elimination, and then use F_{32X} and F_{32Y} in the decoupled solution (5) from the link 2 FBD (identical to the four-bar).

Table II. Number of Multiplications/Divisions (SC)

Method	Gauss-Jordan
8x8	232
5x5 & link 2	72
Reduction	69%

There is a substantial **69% reduction in computational cost** for Gauss-Jordan elimination with the 5x5 plus decoupled link 2 method compared to the full 8x8 approach in common use. Also, we needn't do unnecessary calculations with the three 5x1 columns of zeros.

3.4 SIX-BAR MECHANISMS DECOUPLING

The matrix method may be applied to more complex mechanisms with greater numbers of moving links. The inverse dynamics problem for any mechanism with a dyad of binary links may be partially decoupled in a manner similar to the four-bar and slider-crank mechanisms above.

For example, consider the five well-known Watt and Stevenson planar six-bar mechanisms. All five have 15x15 matrices for inverse dynamics (5 moving links times 3 equations per link; 7 vector internal force unknowns plus 1 torque unknown). Four of these (Stephenson I, Stephenson III – see Figure 3, Watt I, and Watt II) have a dyad of binary links. If these moving links are numbered 5 and 6, one can always solve the inverse dynamics of the FBDs 5 and 6 independently of the other links (six equations in six unknowns). The original 15x15 matrix has nine 6x1 columns of zeros (columns 1, 2, 3, 4, 5, 6, 7, 8, 15) in the six rows 10 through 15, from which a 6x6 subsystem of linear equations may be extracted. Solve for these six unknown vector forces and transfer some of them to the remaining FBDs through equal-and-opposite vector forces.

Next, consider the FBDs of only links 3 and 4: this is 6 equations in 6 unknowns, partially decoupled from the original 15x15 matrix with seven 6x1 columns of zeros (columns 1, 2,

11, 12, 13, 14, 15) in the six rows 4 through 9, and realizing that two of the scalar unknowns are now known from the first decoupling step. Solve six more unknowns from this step and transfer any more equal-and-opposite vector forces to the link 2 FBD. Then the link 2 FBD is solved analogously to (5), i.e. decoupled and analytically, without need for the last 3x3 matrix solution.

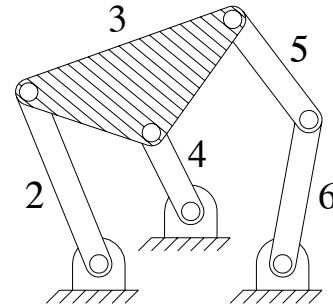


Figure 3. Stephenson III Six-Bar Mechanism

The partial decoupling method is similar and the computational complexity identical for the Stephenson I, Stephenson III, Watt I, and Watt II six-bar mechanisms (Table III). See [9] for details on the Watt II mechanism inverse dynamics.

Table III. Number of Multiplications/Divisions (Six-Bars)

Method	Gauss-Jordan
15x15	1345
6x6 twice & link 2	183
Reduction	86%

There is an astonishing **86% reduction in computational cost** for Gauss-Jordan elimination with the 6x6 twice, plus decoupled link 2 method compared to the full 15x15 approach in common use. Also, we needn't do unnecessary calculations with the sixteen 6x1 columns of zeros.

The Stephenson II six-bar mechanism (Figure 4) does not have a dyad of binary links, thus the above method does not apply. However, upon inspection of the original 15x15 matrix, we find three 12x1 columns of zeros (columns 1, 2, 15) in the twelve rows 4 through 15. This means links 3, 4, 5, and 6 can be solved first independently of link 2, by extracting the appropriate 12x12 matrix equation followed by the standard link 2 solution (5) after making use of an equal-and-opposite vector force that was solved in the first step. The computational savings is not as impressive as in the former six-bar cases: for the Stephenson II six-bar mechanism there is a **46% reduction in computational cost** for Gauss-Jordan elimination with the 12x12 plus decoupled link 2 method compared to the full 15x15 approach in common use.

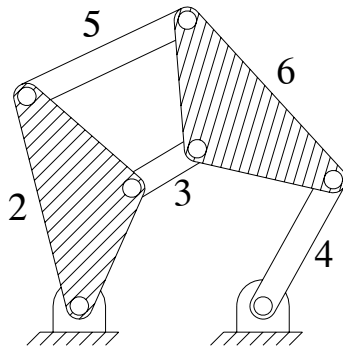


Figure 4. Stephenson II Six-Bar Mechanism

3.5 UICKER SOLUTION

Now, we have stated that kinematics & dynamics textbooks [1-7] have overlooked this partial decoupling method for inverse dynamics using the matrix method – this is a true statement. We discovered this independently and thought it was original. However, Uicker et al. [10] hint at the possible decoupling in their discussion of four-bar mechanism inverse dynamics. They note that links 3 and 4 may be addressed first. However, they do not present the matrix method at all – instead they further find convenient moment centers to make various unknowns disappear until they solve all nine unknowns, one or two at a time. Since our method involves the power and generality of the matrix method we contend it is original.

3.6 SERIAL ROBOT SOLUTION

This paper applies only to closed-chain mechanisms and parallel manipulators. The inverse dynamics problem for open-chain serial manipulators already has the well-known Newton-Euler recursive solution approach [11] which is decoupled link-by-link and needs no matrix techniques.

4. CONCLUSION

This paper presented a method to partially decouple the solutions to inverse dynamics of common planar mechanisms using the matrix method, leading to substantial savings in computational cost. The kinematics & dynamics textbooks have overlooked this significant simplification in the matrix method. Now, some may question the need for this method due to fast PC processors and cheap and plentiful memory. We contend that engineers must always consider computational efficiency, no matter how fast and powerful computers become in the future.

ACKNOWLEDGEMENTS

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