

A Brief Review of Matrices and Linear Algebra

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This document is intended as a reference guide to help students review matrices and linear algebra for use in kinematics, dynamics, controls, biomechanics, and robotics. The usefulness of this document extends well beyond these fields. However, it IS NOT intended to replace a textbook in this field of mathematics.

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Matrix Definition

A matrix is an $m \times n$ array of numbers, where m is the number of rows and n is the number of columns.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrices may be used to simplify and standardize the solution of n linear equations in n unknowns (where $m = n$). Matrices are used in velocity, acceleration, and dynamics linear equations (matrices are not used in analytical position analysis, which requires a non-linear solution).

Special Matrices

These are demonstrated for 3x3 matrices, but apply to all matrix sizes.

square matrix ($m = n = 3$)

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

diagonal matrix

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

identity matrix

$$[I_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

transpose matrix

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad (\text{switch rows \& columns})$$

symmetric matrix

$$[A] = [A]^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

column vector (3x1 matrix)

$$\{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

row vector (1x3 matrix)

$$\{X\}^T = \{x_1 \quad x_2 \quad x_3\}$$

Matrix Operations

Matrix Addition

add like terms and keep the results in place

$$[C] = [A] + [B]$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Matrix Multiplication with a Scalar

multiply each matrix term by the scalar k and keep the results in place

$$k[A] = k \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Matrix Multiplication

$$[C] = [A][B]$$

In general, $[A][B] \neq [B][A]$

The row and column indices must line up as follows.

$$[C] = [A][B]$$

$$(m \times n) \equiv (m \times p)(p \times n)$$

That is, in a matrix multiplication product, the number of columns p in the left-hand matrix must equal the number of rows p in the right-hand matrix. If this condition is not met, the matrix multiplication is undefined and cannot be done.

The size of the resulting matrix $[C]$ is from the number of rows m of the left-hand matrix and the number of columns n of the right-hand matrix, $m \times n$.

Multiplication proceeds by multiplying like terms and adding them, along the rows of the left-hand matrix and down the columns of the right-hand matrix (use your index fingers from the left and right hands for the left-hand rows and right-hand columns, respectively).

Examples:

$$[C] = [A][B]$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{bmatrix}$$

$$(2 \times 1) \equiv (2 \times 2)(2 \times 1)$$

Note the inner indices ($p = 2$) must match, as stated above, and the dimension of the result is dictated by the outer indices, i.e. $m \times n = 2 \times 1$.

For multiplying matrices of multiple columns, follow the same procedure, first using the right index finger down the first column to yield the first column of the result and then using the right index finger down the second column to yield the second column of the result; both use rows 1 and 2.

$$[C] = [A][B]$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$(2 \times 2) \equiv (2 \times 2)(2 \times 2)$$

The inner indices ($p = 2$) must match and the dimension of the result is dictated by the outer indices, i.e. $m \times n = 2 \times 2$.

Another analytical example shows non-square matrix multiplication is possible:

$$[C] = [A][B]$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{bmatrix}$$

$$(2 \times 1) \equiv (2 \times 3)(3 \times 1)$$

The inner indices ($p = 3$) must match and the dimension of the result is dictated by the outer indices, i.e. $m \times n = 2 \times 1$.

We conclude with two strange but legit analytical matrix multiplication examples.

$$\begin{aligned}
 [C] &= [A][B] \\
 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \\
 &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}
 \end{aligned}$$

(1x1) \equiv (1x3)(3x1)

This example is the dot product between two column vectors, i.e. the dot product can be found by transposing the first vector and using matrix multiplication:

$$\{A\} \cdot \{B\} = \{A\}^T \{B\}$$

The last example is even stranger because the matrix multiplication here only involves one multiplication for each term in the result.

$$\begin{aligned}
 [C] &= [A][B] \\
 &= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} \\
 &= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} \end{bmatrix}
 \end{aligned}$$

(3x3) \equiv (3x1)(1x3)

This happens whenever the number of columns in the left-hand matrix equals the number of rows in the right-hand matrix, and both are $p = 1$.

Numerical Matrix Multiplication Examples

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad [B] = \begin{bmatrix} 7 & 8 \\ 9 & 8 \\ 7 & 6 \end{bmatrix}$$

$$[C] = [A][B]$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 8 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7+18+21 & 8+16+18 \\ 28+45+42 & 32+40+36 \end{bmatrix} = \begin{bmatrix} 46 & 42 \\ 115 & 108 \end{bmatrix}$$

$$(2 \times 2) \equiv (2 \times 3)(3 \times 2)$$

$$[D] = [B][A]$$

$$= \begin{bmatrix} 7 & 8 \\ 9 & 8 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7+32 & 14+40 & 21+48 \\ 9+32 & 18+40 & 27+48 \\ 7+24 & 14+30 & 21+36 \end{bmatrix} = \begin{bmatrix} 39 & 54 & 69 \\ 41 & 58 & 75 \\ 31 & 44 & 57 \end{bmatrix}$$

$$(3 \times 3) \equiv (3 \times 2)(2 \times 3)$$

Note in this example that $[A][B] \neq [B][A]$; they are not even of the same size.

Matrix Determinant

The determinant of a square $n \times n$ matrix is a scalar. The matrix determinant is undefined for a non-square matrix. The determinant of a square matrix A is denoted $\det(A)$ or $|A|$. The determinant notation should not be confused with the absolute-value symbol. The MATLAB function for matrix determinant is **det(A)**.

If a nonhomogeneous system of n linear equations in n unknowns is dependent, the coefficient matrix A is singular, and the determinant of matrix A is zero. In this case no unique solution exists to these equations. On the other hand, if the matrix determinant is non-zero, then the matrix is non-singular, the system of equations is independent, and a unique solution exists.

The formula to calculate a 2×2 matrix determinant is straight-forward.

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

To calculate the determinant of 3×3 and larger square matrices, we can expand about any one row or column, utilizing sub-matrix determinants. Each sub-determinant is formed by crossing out the current row and its column and retaining the remaining terms as an $n-1 \times n-1$ square matrix, each of whose determinants must also be evaluated in the process. The pivot term (the entry in the cross-out row and column) multiplies the sub-matrix determinants, and there is an alternating $+/-/+/-$ etc. sign pattern. Here is an explicit example for a 3×3 matrix, expanding about the first row (all other options will yield identical results).

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = +a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= +a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

For a 3 x 3 matrix only, the determinant can alternatively be calculated as shown below, by copying columns 1 and 2 outside the matrix, multiplying the downward diagonals with + signs and multiplying the upward diagonals with – signs (the result is the same as in the above formula).

$$|A| = \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & k & g & h \end{vmatrix}$$

$$= aek + bfg + cdh - gec - hfa - kdb = a(ek - hf) - b(kd - fg) + c(dh - ge)$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{23}a_{31} - a_{33}a_{21}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

A common usage of the 3 x 3 matrix determinant is to calculate the cross product $\underline{P}_1 \times \underline{P}_2$.

$$\underline{P}_1 \times \underline{P}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_{1x} & p_{1y} & p_{1z} \\ p_{2x} & p_{2y} & p_{2z} \end{vmatrix} = +\hat{i} \begin{vmatrix} p_{1y} & p_{1z} \\ p_{2y} & p_{2z} \end{vmatrix} - \hat{j} \begin{vmatrix} p_{1x} & p_{1z} \\ p_{2x} & p_{2z} \end{vmatrix} + \hat{k} \begin{vmatrix} p_{1x} & p_{1y} \\ p_{2x} & p_{2y} \end{vmatrix} = \begin{pmatrix} p_{1y}p_{2z} - p_{1z}p_{2y} \\ p_{1x}p_{2z} - p_{1z}p_{2x} \\ p_{1x}p_{2y} - p_{1y}p_{2x} \end{pmatrix}$$

Matrix Inversion

Since we cannot divide by a matrix, we multiply by the matrix inverse instead. Given $[C] = [A][B]$, solve for $[B]$.

$$[C] = [A][B] \Rightarrow$$

$$[A]^{-1}[C] = [A]^{-1}[A][B]$$

$$= [I][B]$$

$$= [B]$$

$$\Rightarrow [B] = [A]^{-1}[C]$$

Matrix $[A]$ must be square ($m = n$) to invert. The following math facts must be true:

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

where $[I]$ is the identity matrix, the matrix 1 (ones on the diagonal and zeros everywhere else). To calculate the matrix inverse use the following expression.

$$[A]^{-1} = \frac{[\text{adjoint}(A)]}{|A|}$$

where $|A|$ is the determinant of $[A]$.

$$[\text{adjoint}(A)] = [\text{cofactor}(A)]^T$$

$$\text{cofactor}(A) \quad a_{ij} = (-1)^{i+j} |A_{ij}|$$

minor $|A_{ij}|$ is the determinant of the submatrix of $[A]$ with row i and column j removed.

For another example, given $[C] = [A][B]$, solve for $[A]$

$$[C] = [A][B] \Rightarrow$$

$$[C][B]^{-1} = [A][B][B]^{-1}$$

$$= [A][I]$$

$$= [A]$$

$$\Rightarrow [A] = [C][B]^{-1}$$

In general the order of matrix multiplication and inversion is crucial and cannot be changed.

The MATLAB function for matrix determinant is **inv(A)**.

2x2 Matrix Inversion

For a 2x2 matrix only, the above method to calculate the matrix inverse based on the adjoint and determinant yields a simple formula:

The inverse of matrix [A]:

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is:

$$[A]^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

That is, simply switch the main diagonal terms and negate the off-diagonal terms, then divide by the determinant $|A| = a_{11}a_{22} - a_{21}a_{12}$.

Check: $[A][A]^{-1}$ (also $[A]^{-1}[A]$) must equal to the 2x2 Identity matrix ;

$$\begin{aligned} [A][A]^{-1} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \\ &= \frac{1}{|A|} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \\ &= \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & -a_{11}a_{12} + a_{12}a_{11} \\ a_{21}a_{22} - a_{22}a_{21} & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Q.E.D.

The reader is left to verify that the commuted case also works, i.e. $[A]^{-1}[A] = [I_2]$.

Solving a System of Linear Equations

We can solve n linear equations in n unknowns with the help of a matrix. Below is an example for $n = 3$.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Where a_{ij} are the nine known numerical equation coefficients, x_i are the three unknowns, and b_i are the three known right-hand-side terms. Using matrix multiplication backwards, this is written as $[A]\{x\} = \{b\}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

where:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{is the matrix of known numerical coefficients}$$

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \text{is the vector of unknowns to be solved and}$$

$$\{b\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \quad \text{is the vector of known numerical right-hand-side terms.}$$

There is a unique solution:

$$\{x\} = [A]^{-1} \{b\}$$

only if $[A]$ has full rank. If not, $|A| = 0$ (the determinant of coefficient matrix $[A]$ is zero) and the inverse of matrix $[A]$ is undefined (since it would require dividing by zero; in this case the rank is not full, it is less than 3, which means not all rows/columns of $[A]$ are linearly independent).

Gaussian Elimination is more robust and more computationally efficient than matrix inversion to solve the problem $[A]\{x\} = \{b\}$ for $\{x\}$.

Matrix Example – solve linear equations

Solution of 2x2 coupled linear equations.

$$\begin{array}{l} x_1 + 2x_2 = 5 \\ 6x_1 + 4x_2 = 14 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 14 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix} \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad \{b\} = \begin{Bmatrix} 5 \\ 14 \end{Bmatrix}$$

$$\{x\} = [A]^{-1} \{b\}$$

$$|A| = 1(4) - 2(6) = -8$$

The determinant of $[A]$ is non-zero so there is a unique solution. Using the 2x2 matrix inverse formula from earlier:

$$[A]^{-1} = \frac{1}{|A|} \begin{bmatrix} 4 & -2 \\ -6 & 1 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 4 & -2 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 3/4 & -1/8 \end{bmatrix}$$

check the $[A]^{-1}$ result:

$$[A][A]^{-1} = [A]^{-1}[A] = [I_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{checks;}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 3/4 & -1/8 \end{bmatrix} \begin{Bmatrix} 5 \\ 14 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \quad \text{answer.}$$

Check this solution by substituting the answer $\{x\}$ into the original equations $[A]\{x\} = \{b\}$ and using matrix multiplication to ensure the required original $\{b\}$ results.

$$\begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 1(1) + 2(2) \\ 6(1) + 4(2) \end{Bmatrix} = \begin{Bmatrix} 5 \\ 14 \end{Bmatrix} \quad \text{checks.}$$

Matrix Examples in MATLAB

```

%-----
% Matrices.m - matrix examples
%           Dr. Bob, ME 3011
%-----

clear; clc;

A1 = diag([1 2 3])      % 3x3 diagonal matrix
A2 = eye(3)             % 3x3 identity matrix

A3 = [1 2;3 4];        % matrix addition
A4 = [5 6;7 8];
Add = A3 + A4

k = 10;                % matrix-scalar multiplication
MultSca = k*A3

Trans = A4'            % matrix transpose (swap rows and columns)

A5 = [1 2 3;4 5 6];    % define two matrices
A6 = [7 8;9 8;7 6];
A7 = A5*A6              % matrix-matrix multiplication
A8 = A6*A5

A9 = [1 2;6 4];        % matrix for linear equations solution
b = [5;14];            % define RHS vector
dA9 = det(A9)          % calculate determinant of A
invA9 = inv(A9)        % calculate the inverse of A
x = invA9*b            % solve linear equations
x1 = x(1);             % extract answers
x2 = x(2);
Check = A9*x           % check answer - should be b
xG = A9\b              % Gaussian elimination is more efficient

who                    % display the user-created variables
whos                   % user-created variables with dimensions

```

The first solution of the linear equations above uses the matrix inverse. To solve linear equations, **Gaussian Elimination** is more efficient (more on this in the dynamics notes later) and more robust numerically; Gaussian elimination implementation is given in the third to the last line of the above m-file (with the back-slash).

Since the equations are linear, there is a unique solution (assuming the equations are linearly independent, i.e. the matrix is not near a singularity) and so both solution methods will yield the same answer.

Output of Matrices.m

A1 =

1	0	0
0	2	0
0	0	3

A2 =

1	0	0
0	1	0
0	0	1

Add =

6	8
10	12

MultSca =

10	20
30	40

Trans =

5	7
6	8

A7 =

46	42
115	108

A8 =

39	54	69
41	58	75
31	44	57

dA9 = -8

invA9 =

-0.5000	0.2500
0.7500	-0.1250

x =

1
2

Check =

5
14

xG =

1
2