An Atlas of Engineering Dynamic Systems, Models, and Transfer Functions

Dr. Bob Williams, williar4@ohio.edu

Mechanical Engineering, Ohio University

This document presents the models and/or transfer functions of some real-world control systems.

Models are the mathematical descriptions of real-world systems, simplified by various assumptions, ignoring some nonlinear and higher effects. Models are collections of ordinary differential equations and algebraic equations. These equations must be linearized, if necessary, to work in classical controls.

Transfer functions are the mathematical vehicle of classical controls. Transfer functions are defined as the Laplace transform of the output variable divided by the Laplace transform of the input variable, with zero initial conditions. Transfer functions represent the system dynamics, as described by the simplified model – they yield the simulated system output given various inputs. Transfer functions can be derived for the open-loop, closed-loop, and/or smaller system components. Block diagrams are used for graphical representation, where the blocks have transfer functions representing the dynamics of certain system components, while the arrows represent system variables.

The models and transfer functions summarized in this document only give the bottom-line results, without derivations or much explanation. The reader is referred to the various references for more details.

Atlas of Engineering Dynamic Systems, Models, and Transfer Functions

Table of Contents

1. Common System Variables

We start with a table of like variables playing the same role in different engineering systems.

System	**Rate** $r(t)$	**Quantity** $\int r(t)dt$	**Effort** $e(t)$	**Impulse** $\int e(t)dt$
translational mechanical	velocity $v(t)$	displacement $x(t)$	force $f(t)$	impulse
rotational mechanical	angular velocity $\omega(t)$	angular displacement $\theta(t)$	torque $\tau(t)$	angular impulse
electrical	current $i(t)$	charge $q(t)$	voltage $v(t)$	flux $\phi(t)$
incompressible fluid	volume flow rate $q(t)$	volume $V(t)$	pressure $p(t)$	none
compressible fluid	mass flow rate $q_m(t)$	mass $m(t)$	pressure $p(t)$	none
thermal	heat flow rate $q(t)$	heat energy $Q(t)$	temperature $T(t)$	none

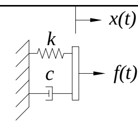
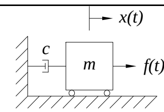
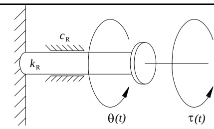
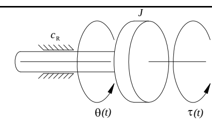
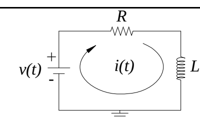
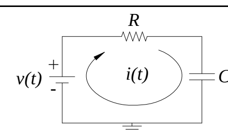
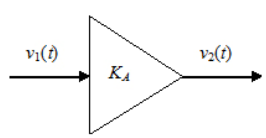
2. Zeroth-Order System Examples

Name	Model	G(s)
gear ratio	$n = \frac{\omega_{IN}(t)}{\omega_{OUT}(t)} = \frac{\theta_{IN}(t)}{\theta_{OUT}(t)} = \frac{\tau_{OUT}(t)}{\tau_{IN}(t)}$	$\frac{\Omega_{OUT}(s)}{\Omega_{IN}(s)} = \frac{\Theta_{OUT}(s)}{\Theta_{IN}(s)} = \frac{1}{n} \quad \frac{\mathrm{T}_{OUT}(s)}{\mathrm{T}_{IN}(s)} = n$
rack and pinion	$l(t) = r\theta(t) \quad \tau(t) = rf(t)$	$\frac{L(s)}{\Theta(s)} = \frac{V(s)}{\Omega(s)} = r \quad \frac{F(s)}{\mathrm{T}(s)} = \frac{1}{r}$
Hooke's Law	$f(t) = kx(t) \quad \tau(t) = k_R\theta(t)$	$\frac{F(s)}{X(s)} = k \quad \frac{T(s)}{\Theta(s)} = k_R$
series / parallel springs	$x(t) = \left[\frac{1}{k_1} + \frac{1}{k_2}\right] f(t) \quad f(t) = (k_1 + k_2)x(t)$	$\frac{F(s)}{X(s)} = \frac{k_1 k_2}{k_1 + k_2} \quad \frac{F(s)}{X(s)} = k_1 + k_2$
viscous damping	$f(t) = cv(t) \quad \tau(t) = c_R\omega(t)$	$\frac{F(s)}{V(s)} = c \quad \frac{T(s)}{\Omega(s)} = c_R$
Newton's Second Law Euler's Rotational Law	$f(t) = ma(t) \quad \tau(t) = J\alpha(t)$	$\frac{A(s)}{F(s)} = \frac{1}{m} \quad \frac{\mathrm{A}(s)}{\mathrm{T}(s)} = \frac{1}{J}$
accelerometer, low-frequency (Dorf & Bishop)	$(k/m)x(t) = \omega^2 x_{IN}(t)$	$\frac{X(s)}{X_{IN}(s)} = \frac{\omega^2}{k/m}$
motor torque	$\tau(t) = K_T i(t)$	$\frac{T(s)}{I(s)} = K_T$
back emf	$v_B(t) = K_B\omega_M(t)$	$\frac{V_B(s)}{\Omega_M(s)} = K_B$

Zeroth-Order System Examples (continued)

Name	Model	G(s)
capacitor ($q(t)$ ~ charge)	$q(t) = Cv(t)$	$\frac{Q(s)}{V(s)} = C \qquad \frac{V(s)}{Q(s)} = \frac{1}{C}$
resistor	$v(t) = Ri(t)$	$\frac{V(s)}{I(s)} = R \qquad \frac{I(s)}{V(s)} = \frac{1}{R}$
inductor ($\phi(t)$ ~ flux)	$\phi(t) = Li(t)$	$\frac{\Phi(s)}{I(s)} = L \qquad \frac{I(s)}{\Phi(s)} = \frac{1}{L}$
potentiometer	$v_1(t)R_2 = v_2(t)(R_1 + R_2)$	$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R_1 + R_2}$
tachometer	$v(t) = K_t\omega(t)$	$\frac{V(s)}{\Omega(s)} = K_t$
DC amplifier, zero time constant	$v_2(t) = K_A v_1(t)$	$\frac{V_2(s)}{V_1(s)} = K_A$
series / parallel resistors	$v(t) = (R_1 + R_2)i(t) \qquad i(t) = \left[\frac{1}{R_1} + \frac{1}{R_2}\right]v(t)$	$\frac{V(s)}{I(s)} = R_1 + R_2 \qquad \frac{V(s)}{I(s)} = \frac{R_1 R_2}{R_1 + R_2}$

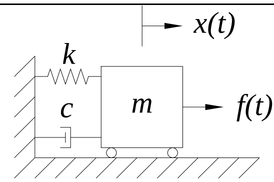
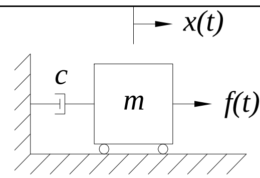
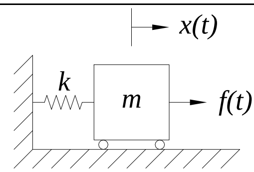
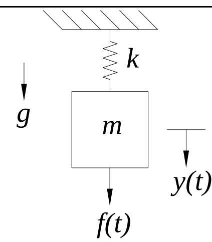
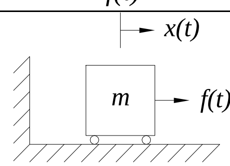
3. First-Order System Examples

Name	Diagram	Model	*G*(*s*)	τ
massless translational mechanical system		$c\dot{x}(t)+kx(t)=f(t)$	$\frac{X(s)}{F(s)}=\frac{1}{cs+k}$	$\frac{c}{k}$
springless translational mechanical system		$m\dot{v}(t)+cv(t)=f(t)$	$\frac{V(s)}{F(s)}=\frac{1}{ms+c}$	$\frac{m}{c}$
inertialess rotational mechanical system		$c_R\dot{\theta}(t)+k_R\theta(t)=\tau(t)$	$\frac{\Theta(s)}{T(s)}=\frac{1}{c_Rs+k_R}$	$\frac{c_R}{k_R}$
springless rotational mechanical system		$J\dot{\omega}(t)+c_R\omega(t)=\tau(t)$	$\frac{\Omega(s)}{T(s)}=\frac{1}{Js+c_R}$	$\frac{J}{c_R}$
LR series electrical circuit		$L\frac{di(t)}{dt}+Ri(t)=v(t)$	$\frac{I(s)}{V(s)}=\frac{1}{Ls+R}$	$\frac{L}{R}$
RC series electrical circuit		$R\dot{q}(t)+\frac{1}{C}q(t)=v(t)$ $Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$	$\frac{Q(s)}{V(s)}=\frac{C}{RCs+1}$ $\frac{I(s)}{V(s)}=\frac{Cs}{RCs+1}$	RC RC
DC amplifier with time constant		$\tau\dot{v}_2(t)+v_2(t)=K_Av_1(t)$	$\frac{V_2(s)}{V_1(s)}=\frac{K_A}{\tau s+1}$	τ

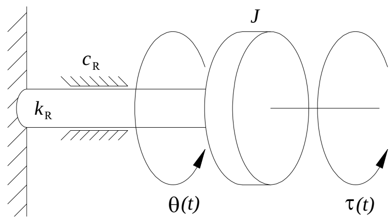
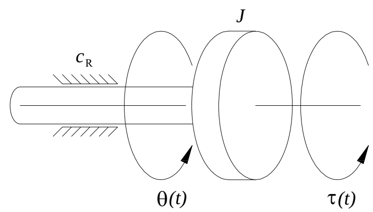
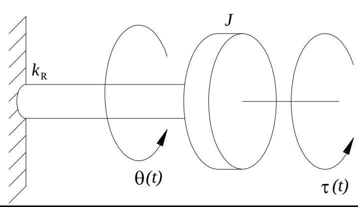
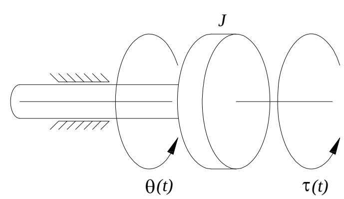
First-Order System Examples (continued)

Name	Model	G(s)
differentiator	$v(t)=\dfrac{dx(t)}{dt}$ $\quad a(t)=\dfrac{dv(t)}{dt}$ $i(t)=\dfrac{dq(t)}{dt}$ $\quad v(t)=\dfrac{d\phi(t)}{dt}$	$\dfrac{V(s)}{X(s)}=s$ $\quad \dfrac{A(s)}{V(s)}=s$ $\dfrac{I(s)}{Q(s)}=s$ $\quad \dfrac{V(s)}{\Phi(s)}=s$
integrator	$x(t)=\int v(t)dt$ $\quad v(t)=\int a(t)dt$ $q(t)=\int i(t)dt$ $\quad \phi(t)=\int v(t)dt$	$\dfrac{X(s)}{V(s)}=\dfrac{1}{s}$ $\quad \dfrac{V(s)}{A(s)}=\dfrac{1}{s}$ $\dfrac{Q(s)}{I(s)}=\dfrac{1}{s}$ $\quad \dfrac{\Phi(s)}{V(s)}=\dfrac{1}{s}$
capacitor	$i(t)=C\dfrac{dv(t)}{dt}$ $\quad v(t)=\dfrac{1}{C}\int i(t)dt$	$\dfrac{I(s)}{V(s)}=Cs$ $\quad \dfrac{V(s)}{I(s)}=\dfrac{1}{Cs}$
resistor	$q(t)=\dfrac{1}{R}\int v(t)dt$ $\quad v(t)=R\dfrac{dq(t)}{dt}$	$\dfrac{Q(s)}{V(s)}=\dfrac{1}{Rs}$ $\quad \dfrac{V(s)}{Q(s)}=Rs$
inductor	$i(t)=\dfrac{1}{L}\int v(t)dt$ $\quad v(t)=L\dfrac{di(t)}{dt}$	$\dfrac{I(s)}{V(s)}=\dfrac{1}{Ls}$ $\quad \dfrac{V(s)}{I(s)}=Ls$
generic sensor	$\tau\dot{y}_{SENS}(t)+y_{SENS}(t)=ky(t)$	$H(s)=\dfrac{Y_{SENS}(s)}{Y(s)}=\dfrac{k}{\tau s+1}$ k gain $\quad$ τ time constant

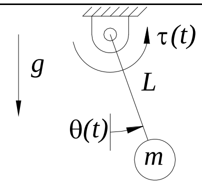
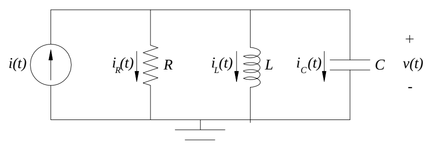
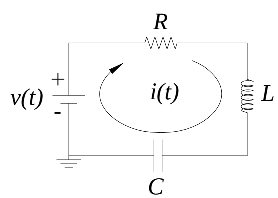
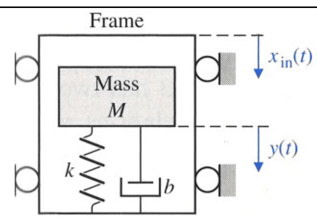
4. Second-Order System Examples

Name	**Diagram**	**Model**	***G*(*s*)**
translational mechanical system		$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=f(t)$ $m\dfrac{dv(t)}{dt}+cv(t)+k\int v(t)dt=f(t)$	$\dfrac{X(s)}{F(s)}=\dfrac{1}{ms^2+cs+k}$ $\dfrac{V(s)}{F(s)}=\dfrac{s}{ms^2+cs+k}$
springless translational mechanical system		$m\ddot{x}(t)+c\dot{x}(t)=f(t)$	$\dfrac{X(s)}{F(s)}=\dfrac{1}{s(ms+c)}$
damperless translational mechanical system		$m\ddot{x}(t)+kx(t)=f(t)$	$\dfrac{X(s)}{F(s)}=\dfrac{1}{ms^2+k}$
damperless translational mechanical system, vertical		$m\ddot{y}(t)+ky(t)=f(t)$	$\dfrac{Y(s)}{F(s)}=\dfrac{1}{ms^2+k}$
mass-only translational mechanical system		$m\ddot{x}(t)=f(t)$	$\dfrac{X(s)}{F(s)}=\dfrac{1}{ms^2}$

Second-Order System Examples (continued)

Name	Diagram	Model	G(s)
rotational mechanical system		$J\ddot{\theta}(t)+c_R\dot{\theta}(t)+k_R\theta(t)=\tau(t)$	$\frac{\Theta(s)}{T(s)}=\frac{1}{Js^2+c_Rs+k_R}$
springless rotational mechanical system		$J\ddot{\theta}(t)+c_R\dot{\theta}(t)=\tau(t)$	$\frac{\Theta(s)}{T(s)}=\frac{1}{s(Js+c_R)}$
damperless rotational mechanical system		$J\ddot{\theta}(t)+k_R\theta(t)=\tau(t)$	$\frac{\Theta(s)}{T(s)}=\frac{1}{Js^2+k_R}$
inertia-only rotational mechanical system		$J\ddot{\theta}(t)=\tau(t)$	$\frac{\Theta(s)}{T(s)}=\frac{1}{Js^2}$

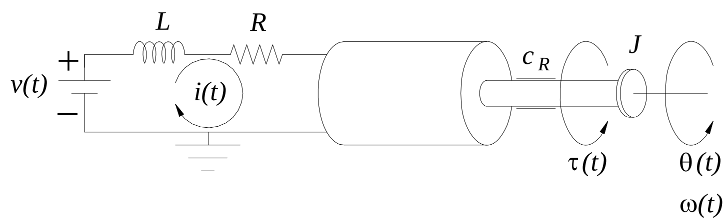
Second-Order System Examples (continued)

Name	Diagram	Model	G(s)
torqued pendulum (linearized)		$\ddot{\theta}(t)+\frac{g}{L}\theta(t)=\frac{1}{mL^2}\tau(t)$	$\frac{\Theta(s)}{\mathrm{T}(s)}=\frac{1}{mL^2\left[s^2+\frac{g}{L}\right]}$
parallel *RLC* circuit		$C\frac{dv(t)}{dt}+\frac{v(t)}{R}+\frac{1}{L}\int v(t)dt=i(t)$ $C\ddot{\phi}(t)+\frac{1}{R}\dot{\phi}(t)+\frac{1}{L}\phi(t)=i(t)$	$\frac{V(s)}{I(s)}=\frac{RLs}{CRLs^2+Ls+R}$ $\frac{\Phi(s)}{I(s)}=\frac{RL}{CRLs^2+Ls+R}$
series *RLC* circuit		$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$ $L\ddot{q}(t)+R\dot{q}(t)+\frac{1}{C}q(t)=v(t)$	$\frac{I(s)}{V(s)}=\frac{Cs}{LCs^2+RCs+1}$ $\frac{Q(s)}{V(s)}=\frac{C}{LCs^2+RCs+1}$
accelerometer (Dorf & Bishop)		$m\ddot{x}(t)+b\dot{x}(t)+kx(t)=-m\ddot{x}_{IN}(t)$	$\frac{X(s)}{X_{IN}(s)}=\frac{-s^2}{s^2+(b/m)s+(k/m)}$
double differentiator		$a(t)=\frac{d^2x(t)}{dt^2}$	$\frac{A(s)}{X(s)}=s^2$
double integrator		$x(t)=\int\int a(t)dt$	$\frac{X(s)}{A(s)}=\frac{1}{s^2}$

5. Real-World Transfer Functions

Simplified DC servomotor (ignoring back emf and inductor)

The figure below shows a simple diagram for deriving the model of a DC servomotor, which is a rotational electromechanical system. On the circuit side, $v(t)$ is the input armature voltage, L is the inductance constant, R is the resistance constant, and $i(t)$ is the armature circuit current. On the rotational mechanical side, J is the lumped rotational inertia of the motor shaft and load, c_R is the rotational viscous damping coefficient, and the output is angular displacement $\theta(t)$ (whose time derivative is angular velocity $\omega(t)$).



From an earlier derivation, the RL series circuit model is:

$$L\frac{di(t)}{dt} + Ri(t) = v(t)$$

where we have ignored the motor back emf voltage. Usually the time constant for the electrical system is much smaller than the time constant for the rotational mechanical system, which means that the electrical system current $i(t)$ rises much faster than the mechanical displacement $\theta(t)$. Therefore, we can ignore the circuit dynamics ($L \approx 0$), so the electrical circuit model simplifies to $Ri(t) = v(t)$, which is simply Ohm's Law.

In a DC servomotor, the generated motor torque is proportional to the circuit current, a linear proportional relationship that holds good for nearly the entire range of operation of the motor:

$$\tau(t) = K_T i(t)$$

K_T is the motor torque constant, which is stamped on the motor housing, available from the motor manufacturer, or determinable by experiment.

The rotational mechanical system dynamic model is derived from a free-body diagram of the rotating motor shaft, using Euler's rotational dynamics law $\sum M = J\ddot{\theta}(t)$:

$$J\ddot{\theta}(t) + c_R\dot{\theta}(t) = \tau(t)$$

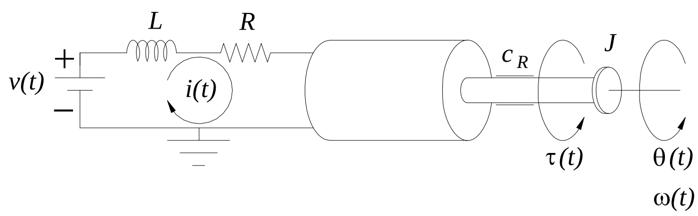
Substituting the electrical models into the rotational mechanical system dynamic model yields:

$$J\ddot{\theta}(t) + c_R\dot{\theta}(t) = \tau(t) = K_T i(t) = \frac{K_T}{R}v(t) \qquad G_\theta(s) = \frac{\Theta(s)}{V(s)} = \frac{K_T/R}{s(Js + c_R)}$$

This is a linear, lumped-parameter, constant-coefficient, second-order ODE. The same model written for angular velocity $\omega(t)$ output is a first-order model:

$$J\dot{\omega}(t) + c_R\omega(t) = \frac{K_T}{R}v(t) \qquad G_\omega(s) = \frac{\Omega(s)}{V(s)} = \frac{K_T/R}{Js + c_R}$$

DC Servomotor[1]



$$J\ddot{\theta}(t) + c_R\dot{\theta}(t) = \tau(t) = K_T i(t)$$

$$L\frac{di(t)}{dt} + Ri(t) = v(t) - v_B(t) = v(t) - K_B\dot{\theta}(t)$$

$$K_T = K_B = K$$

$$G_\omega(s) = \frac{\Omega(s)}{V(s)} = \frac{K}{(Ls+R)(Js+c_R)+K^2}$$

$$G_\theta(s) = \frac{\Theta(s)}{V(s)} = \frac{K}{s\left[(Ls+R)(Js+c_R)+K^2\right]}$$

If we set the armature circuit time constant $\frac{L}{R}$ to zero relative to the mechanical system time constant $\frac{J}{c_R}$ (since the mechanical system dominates), the above transfer functions are simplified to first- and second-order, respectively (rather than the original second- and third-order systems):

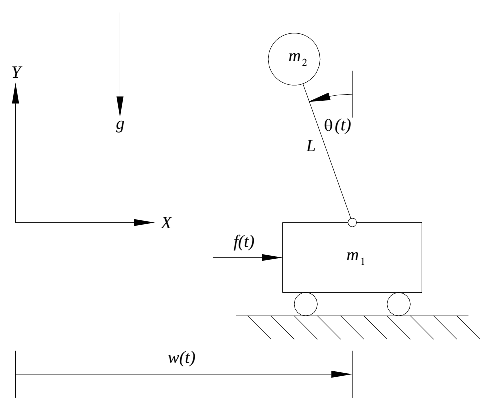
$$G_\omega(s) = \frac{\Omega(s)}{V(s)} = \frac{K}{JRs + (Rc_R + K^2)}$$

$$G_\theta(s) = \frac{\Theta(s)}{V(s)} = \frac{K}{\left[JRs + (Rc_R + K^2)\right]s}$$

[1] R.L. Williams II and D.A. Lawrence, 2007, Linear State-Space Control Systems, John Wiley & Sons, Inc.

suitable for ME 3012 Term Projects

Inverted pendulum[1]



$$
\begin{aligned}
(m_1+m_2)\ddot{w}(t) - m_2 L\cos\theta(t)\ddot{\theta}(t) + m_2 L\sin\theta(t)\dot{\theta}(t)^2 &= f(t) \\
m_2 L^2\ddot{\theta}(t) - m_2 L\cos\theta(t)\ddot{w}(t) - m_2 g L\sin\theta(t) &= 0
\end{aligned}
$$

non-linear

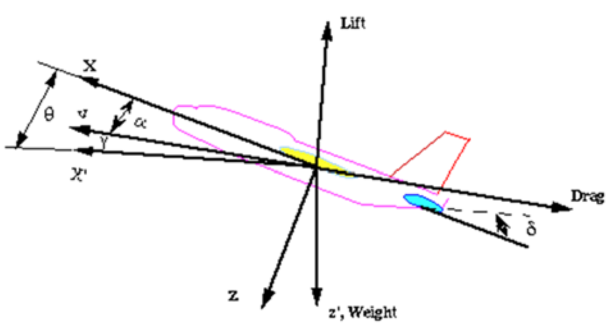
$$
\begin{aligned}
(m_1+m_2)\ddot{w}(t) - m_2 L\ddot{\theta}(t) &= f(t) \\
-m_2\ddot{w}(t) + m_2 L\ddot{\theta}(t) - m_2 g\theta(t) &= 0
\end{aligned}
$$

linearized

In order to derive the overall SISO transfer function for the inverted pendulum, take the Laplace Transform of both sides of both of the linearized ODEs above. Then use algebra to eliminate $W(s)$ between the two equations and arrive at $G(s)$. This process yields the following Type 0, second-order, unstable open-loop transfer function:

$$G(s) = \frac{\Theta(s)}{F(s)} = \frac{1}{m_1 L s^2 - (m_1+m_2)g}$$

Aircraft Pitch Control[2]



$\theta(t)$ output aircraft pitch angle

$\omega(t) = \dot{\theta}(t)$ output aircraft pitch angular velocity

$\delta(t)$ input elevator control angle

$$G(s) = \frac{\Omega(s)}{\Delta(s)} = \frac{1.15s + 0.18}{s^2 + 0.74s + 0.92}$$

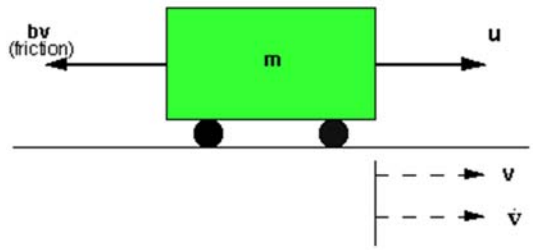
[2] www.engin.umich.edu/group/ctm

Automobile Cruise Control[2]

$$G(s)=\frac{V(s)}{U(s)}=\frac{1}{ms+b}$$

$u(t)$ road force input

$v(t)$ output speed



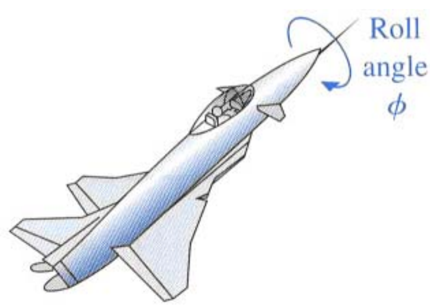
Aircraft Roll Control[3]

$$G(s)=\frac{\Omega(s)}{Q(s)}=\frac{K}{s^2+4s+9}$$

$q(t)$ hydraulic fluid flow

$\omega(t)=\dot{\phi}(t)$ roll velocity

$\phi(t)$ roll (bank) angle

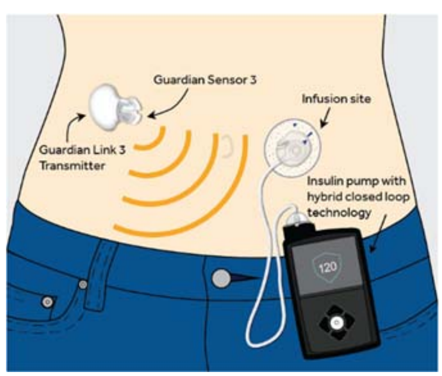


Diabetes Control[3]

$$G(s)=\frac{B(s)}{I(s)}=\frac{s+2}{s(s+1)}$$

$i(t)$ insulin input

$b(t)$ output blood-sugar level



[3] Dorf and Bishop, Modern Control Systems, 11th edition, Pearson Prentice Hall, 2007.

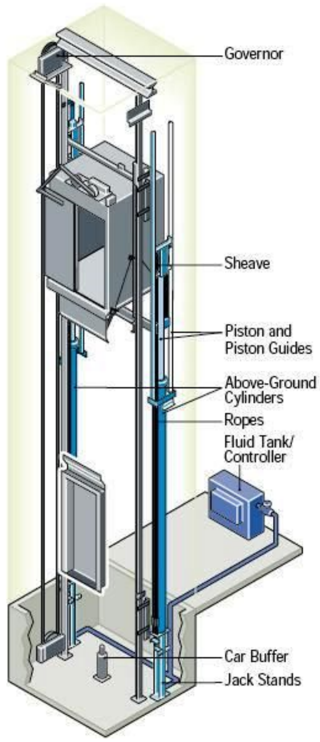
Elevator Control[3]

$$G(s)=\frac{V(s)}{\mathrm{T}(s)}=\frac{1}{s^2+2s+11}$$

$\tau(t)$ torque input

$v(t)=\dot{y}(t)$ elevator velocity

$y(t)$ elevator displacement

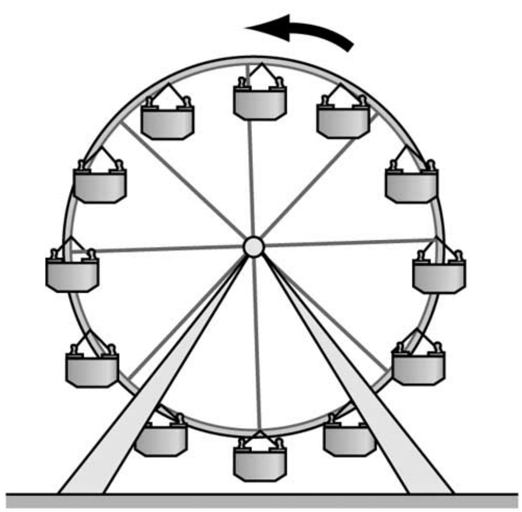


Ferris Wheel Control[3]

$$G(s)=\frac{\Omega(s)}{\mathrm{T}(s)}=\frac{s+6}{(s+2)(s+4)}$$

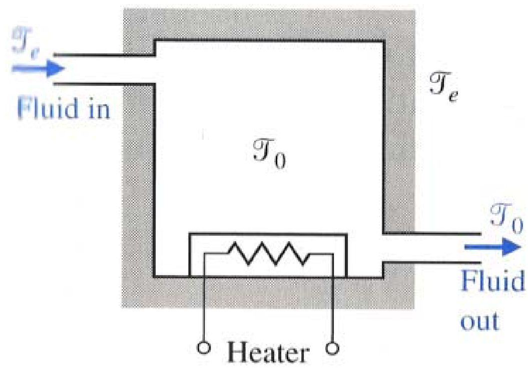
$\tau(t)$ torque input

$\omega(t)$ output angular velocity



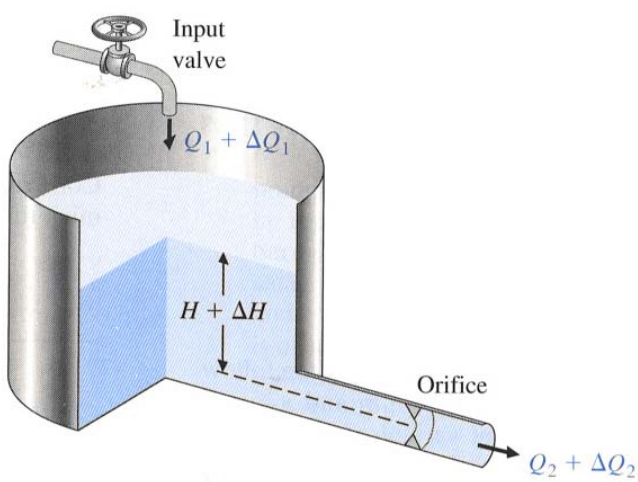
Fluid Heating System[3]

$G(s)=\dfrac{T(s)}{H(s)}=\dfrac{1}{Cs+\left[QW+\dfrac{1}{R}\right]}$ $\tau=\dfrac{RC}{RQW+1}$	• $T(s)$ temperature difference $T_0(s) - T_e(s)$ • C thermal capacitance • Q constant flow rate • W water specific heat • R insulation thermal resistance • $H(s)$ heating element heat flow rate • τ time constant



Fluid Flow Tank System[3]

$G_Q(s)=\dfrac{\Delta Q_2(s)}{\Delta Q_1(s)}=\dfrac{1}{\tau s+1}$ $G_H(s)=\dfrac{\Delta H(s)}{\Delta Q_1(s)}=\dfrac{R}{RCs+1}$	• Q_1 input flow rate • Q_2 output flow rate • H head • τ RC, time constant • R orifice flow resistance • C cross-sectional tank area

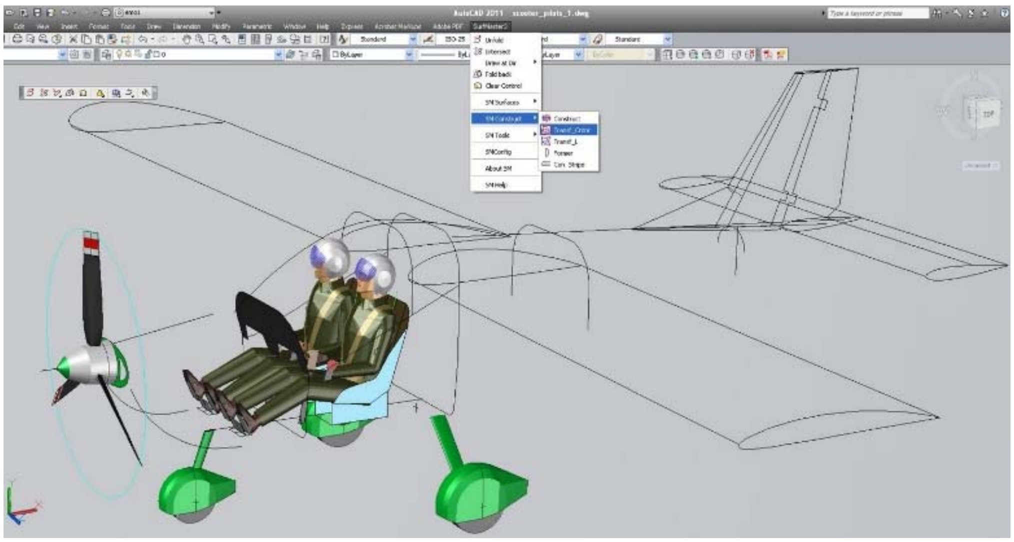


Human "Paper-Pilot" Model[3]

$$G(s)=\frac{\Phi(s)}{\Theta_E(s)}=\frac{-(2s+1)(\tau s-2)}{(0.5s+1)(\tau s+2)}$$

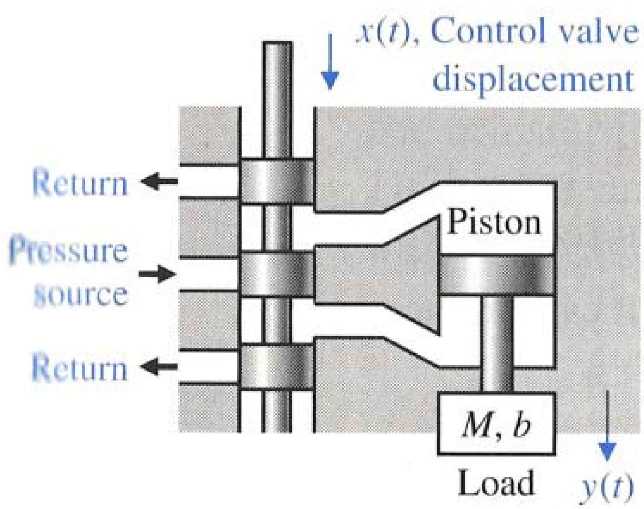
$\theta_E(t)$ input angle error — $\phi(t)$ output elevator angle

$0.25 \le \tau \le 0.50$ — human pilot time constant



Hydraulic Actuator[3]

$G(s)=\dfrac{Y(s)}{X(s)}=\dfrac{K}{s(ms+B)}$	$K=\dfrac{Ak_x}{k_P}$ $\quad B=b+K=\dfrac{A^2}{k_P}$ $k_x=\left.\dfrac{\partial g}{\partial x}\right\|_{x_0}$ $\quad k_P=\left.\dfrac{\partial g}{\partial P}\right\|_{P_0}$	• $g=g(x,P)$ flow • A piston area

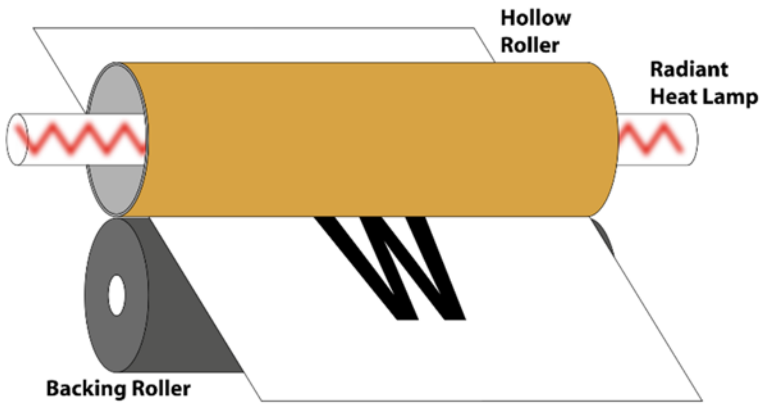


Laser Printer Positioning[3]

$$G(s)=\frac{Y(s)}{\mathrm{T}(s)}=\frac{4(s+50)}{s^2+30s+200}$$

$\tau(t)$ input control torque

$y(t)$ printer head displacement



Paper Processing Tensioning[3]

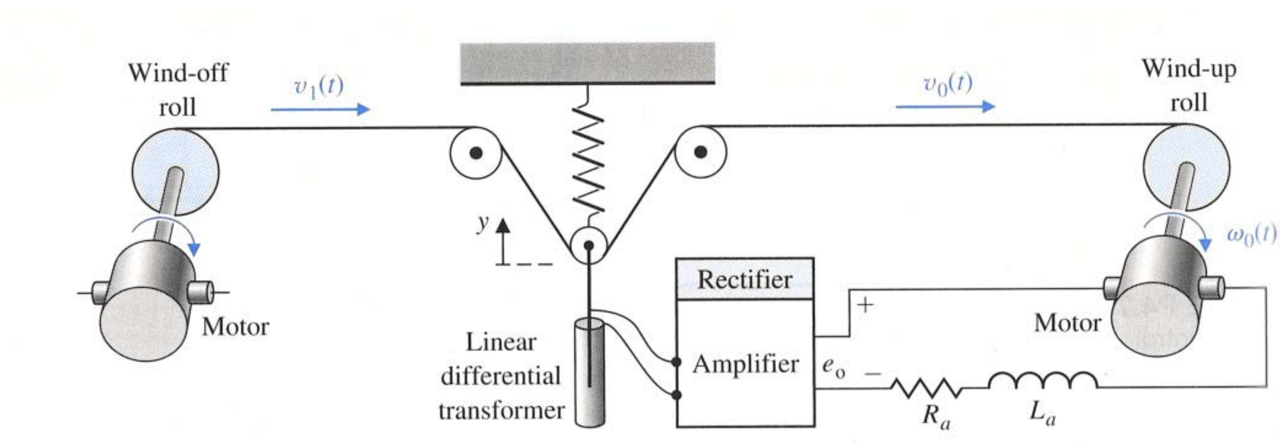
$$G(s)=\frac{\Omega_0(s)}{E_0(s)}=\frac{K_M}{\tau s+1}$$

$e_0(t)$ input motor voltage

$\omega_0(t)$ windup roll velocity

K_M motor constant

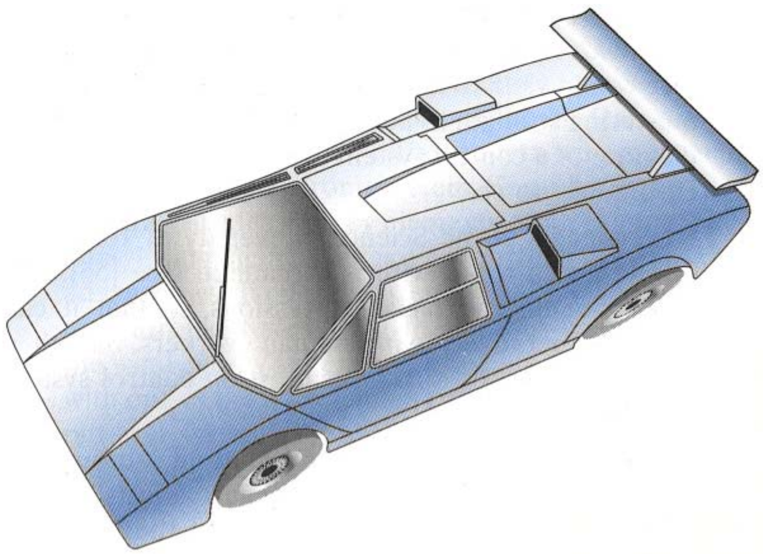
$\tau=\dfrac{L}{R}$ motor time constant



Racecar Speed Control[3]

$$G(s) = \frac{V(s)}{C(s)} = \frac{100}{(s+2)(s+5)}$$

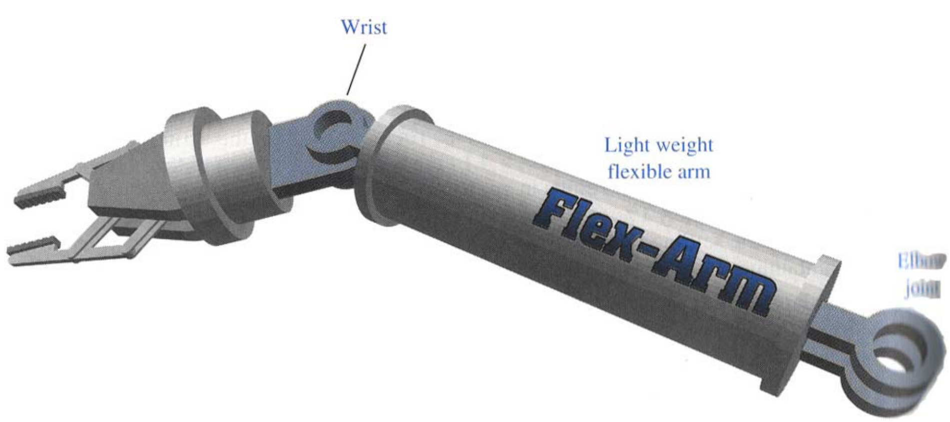
$c(t)$ throttle input $v(t)$ output racecar speed



Robot Elbow Control[3]

$$G(s) = \frac{\Theta(s)}{\mathrm{T}(s)} = \frac{2}{s(s+4)}$$

$\tau(t)$ torque input $\theta(t)$ output elbow angle



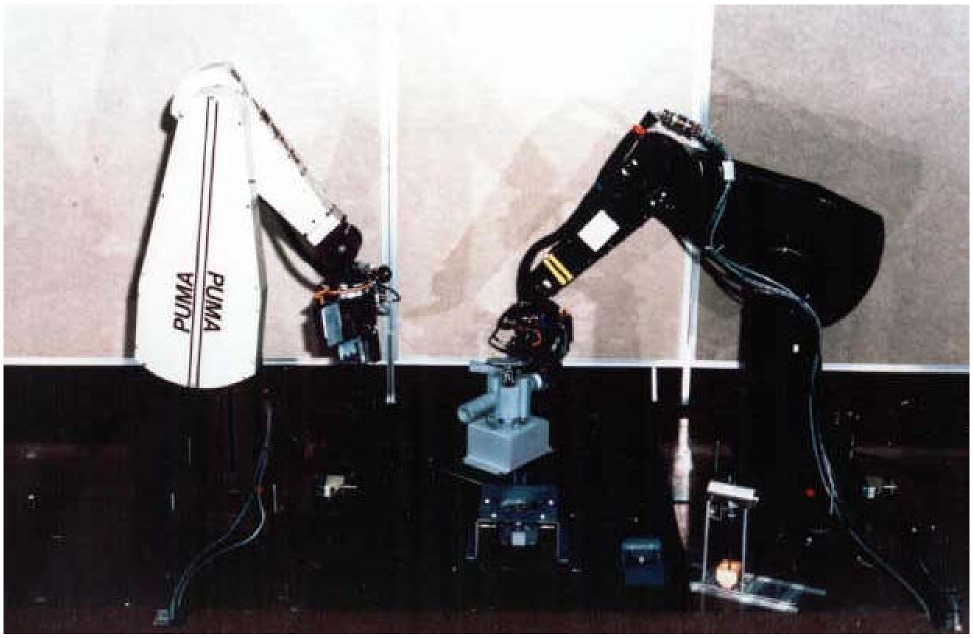
Robot Position Control[3]

$$G(s)=\frac{V(s)}{\mathrm{T}(s)}=\frac{640,000}{s^2+128s+6400}$$

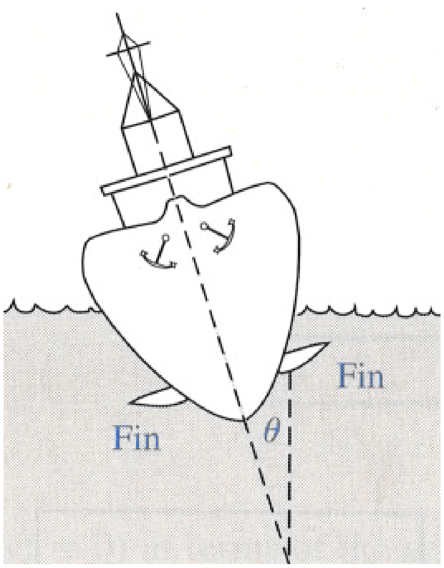
$\tau(t)$ torque input

$v(t)=\dot{y}(t)$ robot velocity

$y(t)$ robot displacement



Ship Stabilization[3]



$$G(s)=\frac{\Theta(s)}{\mathrm{T}_f(s)}=\frac{9}{s^2+1.2s+9}$$

$\tau_f(t)$ fin control input torque

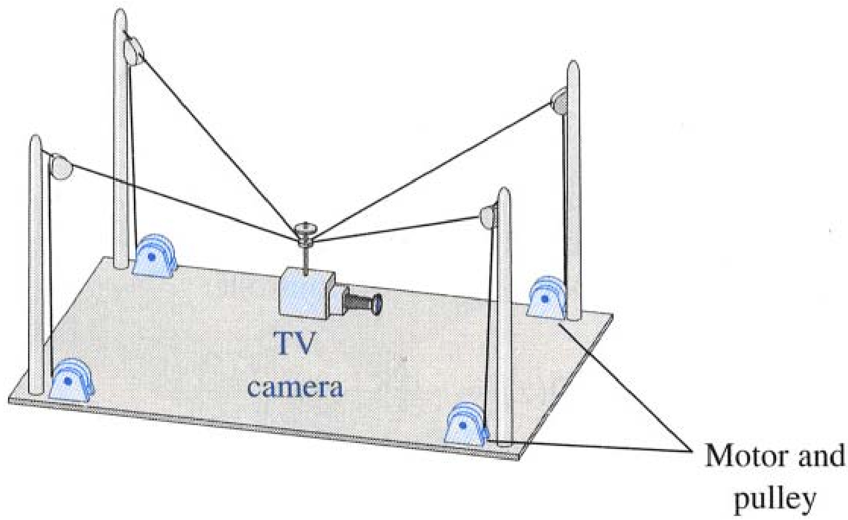
$\theta(t)$ ship output angle

SkyCam Control[3modified]

$$G(s)=\frac{Y(s)}{T(s)}=\frac{1}{s(0.2s+1)}$$

$\tau(t)$ torque input

$y(t)$ output displacement



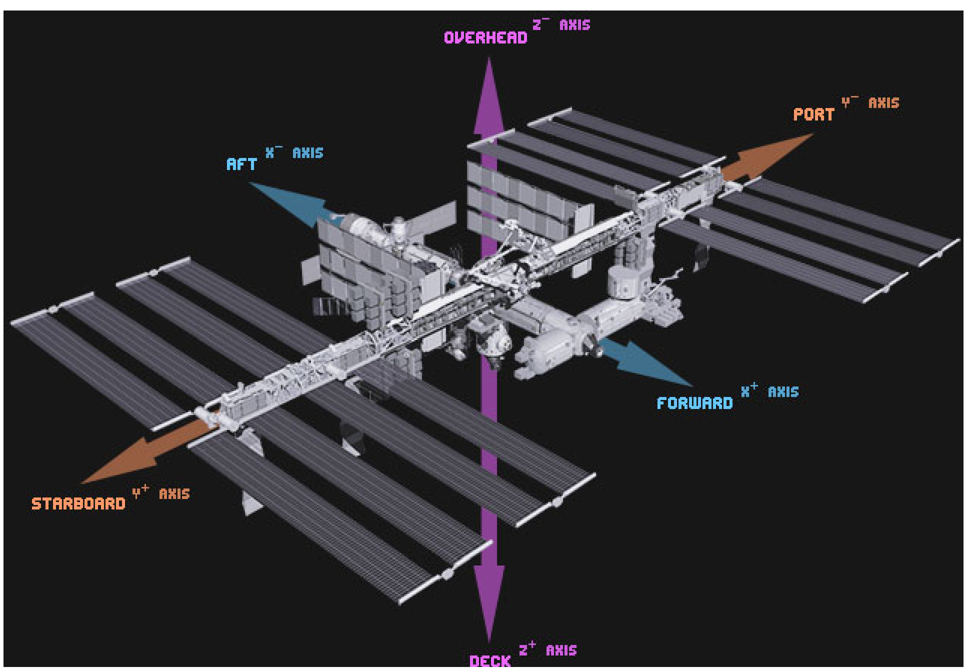
Space Station Orientation Control[3]

$$G(s)=\frac{\Omega(s)}{\mathrm{T}(s)}=\frac{20}{s^2+20s+100}$$

$\tau(t)$ torque input

$\omega(t)=\dot{\phi}(t)$ station velocity

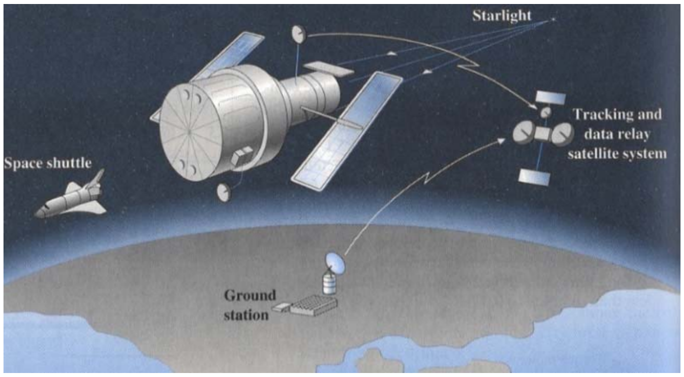
$\phi(t)$ space station angle



Space Telescope Pointing Control[3]

$$G(s)=\frac{\Phi(s)}{T(s)}=\frac{1}{s(s+12)}$$

$\tau(t)$ torque wheel input $\phi(t)$ output telescope angle

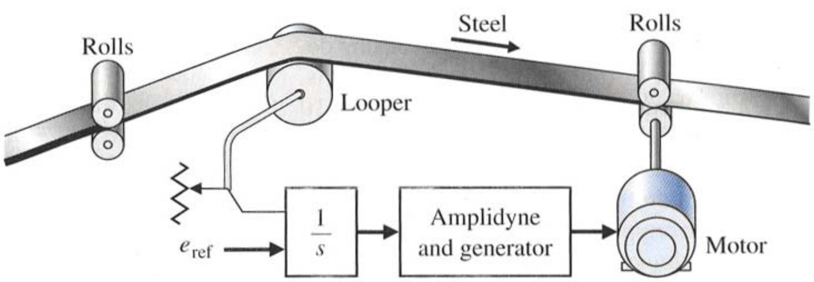


Steel Rolling Thickness Control[3]

$$G(s)=\frac{Y(s)}{\mathrm{T}(s)}=\frac{0.25}{s(s+1)}$$

$\tau(t)$ input motor torque $y(t)$ output steel thickness

$v_0=2000$ ft/min nominal steel speed

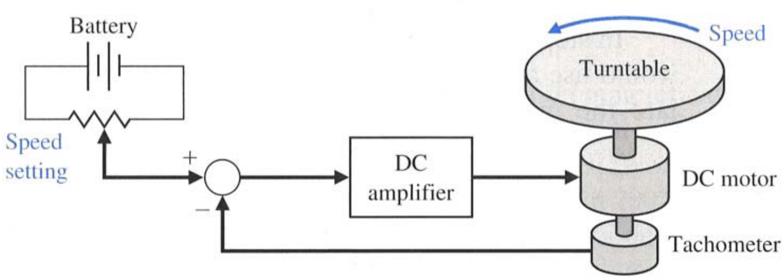


Turntable Angular Speed Control (old-school records)[3]

$$G(s)=\frac{\Omega(s)}{V(s)}=\frac{K_T/R}{Js+b}$$

$v(t)$ input voltage $\omega(t)$ output turntable speed

K_T motor torque constant R circuit resistance



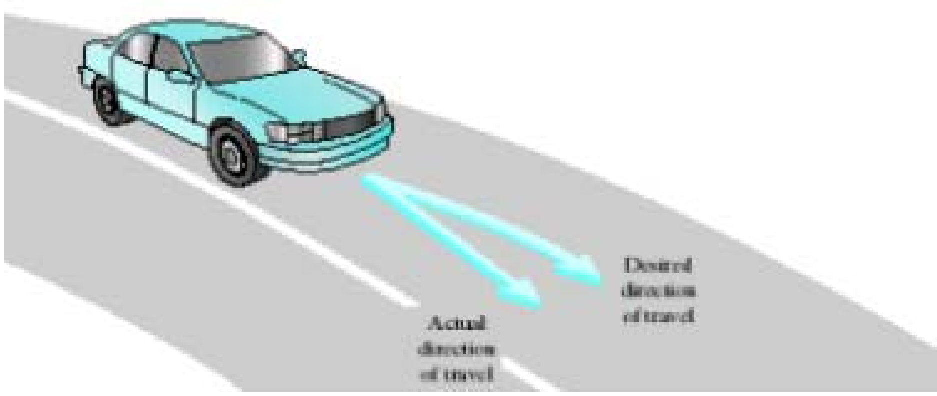
Vehicle Steering Control[3]

$$G(s) = \frac{V(s)}{\Theta(s)} = \frac{1}{s(s+12)}$$

$\theta(t)$ steering wheel angle

$v(t) = \dot{y}(t)$ centerline velocity

$y(t)$ centerline displacement

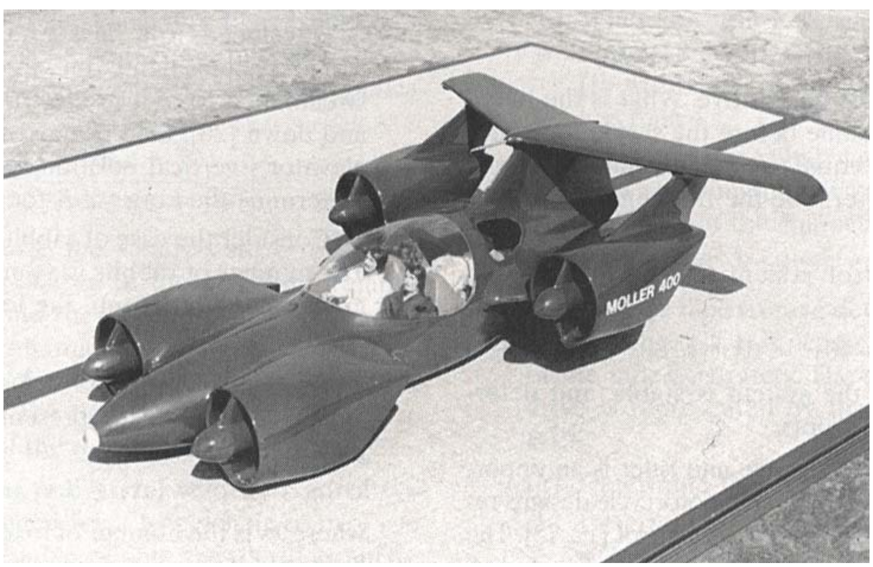


VTOL Aircraft Control[3]

$$G(s) = \frac{Y(s)}{T(s)} = \frac{1}{s(s-1)}$$

$\tau(t)$ thruster input

$y(t)$ vertical displacement

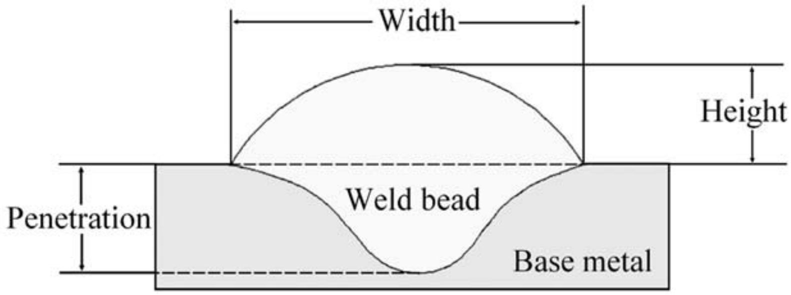


Weld Bead Depth Control[3]

$$G(s) = \frac{Y(s)}{I(s)} = \frac{K}{(0.01s+1)(1.5s+1)}$$

$i(t)$ input current

$y(t)$ output weld bead depth



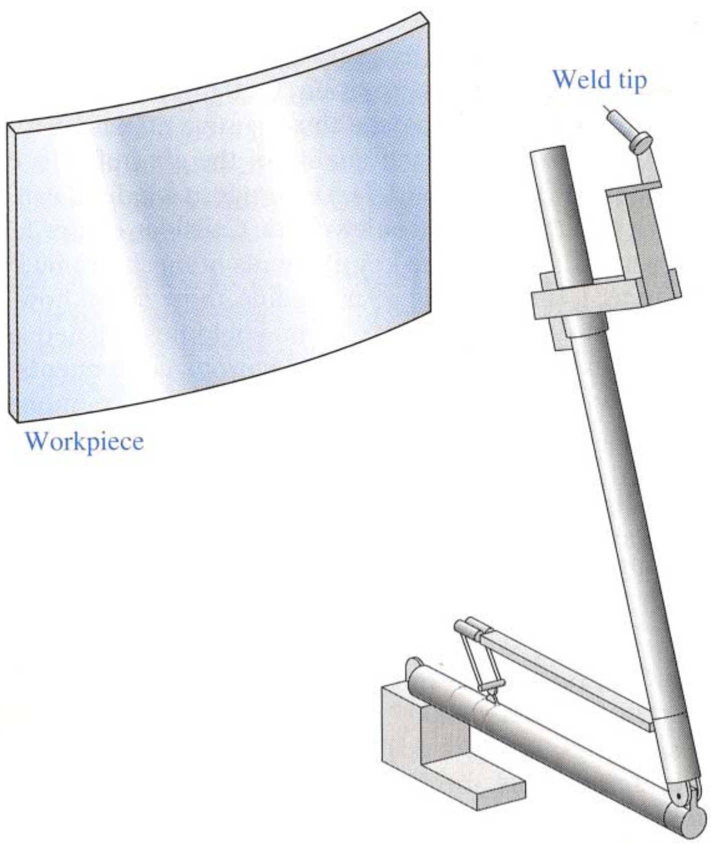
Welding Robot Control[3]

$$G(s) = \frac{V(s)}{\mathrm{T}(s)} = \frac{75(s+1)}{(s+5)(s+20)}$$

$\tau(t)$ torque input

$v(t) = \dot{y}(t)$ robot velocity

$y(t)$ robot displacement



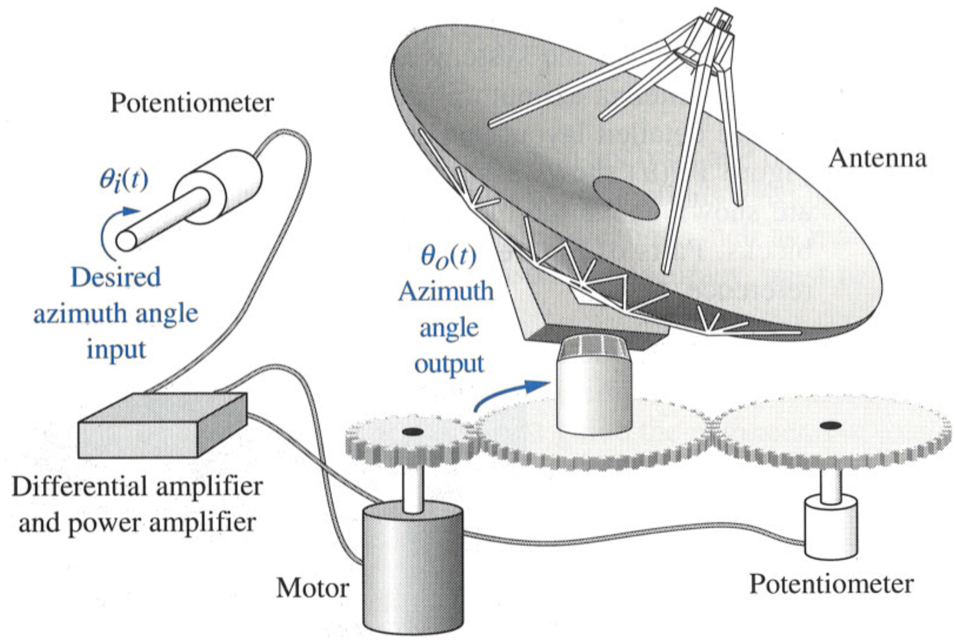
Antenna Azimuth Control[4]

$$G(s) = \frac{\Omega(s)}{V(s)} = \frac{20.83}{(s+100)(s+1.71)}$$

$v(t)$ input voltage

$\omega(t) = \dot{\phi}(t)$ azimuth velocity

$\theta(t)$ azimuth angle

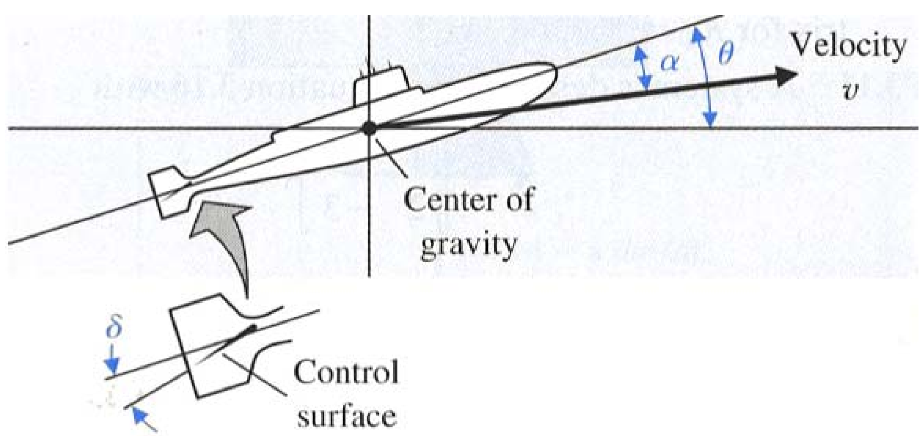


Autonomous Submersible Control[5]

$$G(s) = \frac{\Theta(s)}{\Delta_e(s)} = \frac{-0.13(s+0.44)}{s^2+0.23s+0.02}$$

$\delta_e(t)$ input elevator angle

$\theta(t)$ output pitch angle



---

[4] N.S. Nise, Control Systems Engineering, 2nd edition, Cummings, 1995.
[5] Golnaraghi and Kuo, Automatic Control Systems, 9th edition, Wiley, 2010

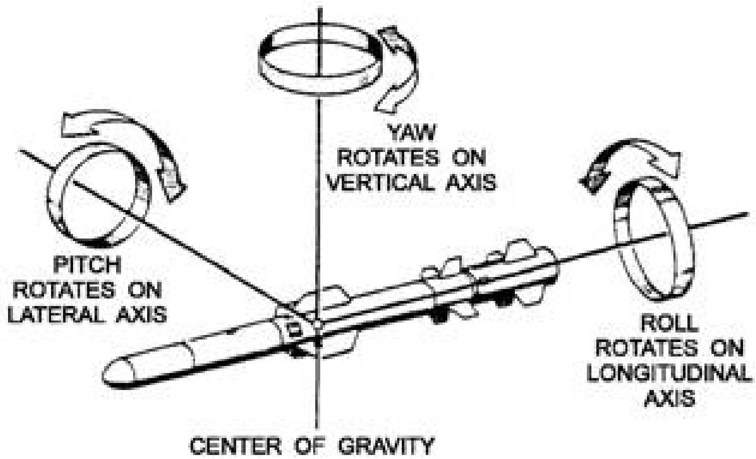
Missile Roll Control[5]

$$G(s)=\frac{P(s)}{\mathrm{E}(s)}=\frac{l_\varepsilon}{s-l_p}=\frac{-l_\varepsilon/l_p}{\tau_a s+1}$$

$\varepsilon(t)$ aileron angle input $p(t)$ output roll rate

$\tau_a=-\dfrac{1}{l_p}$ aerodynamic time constant

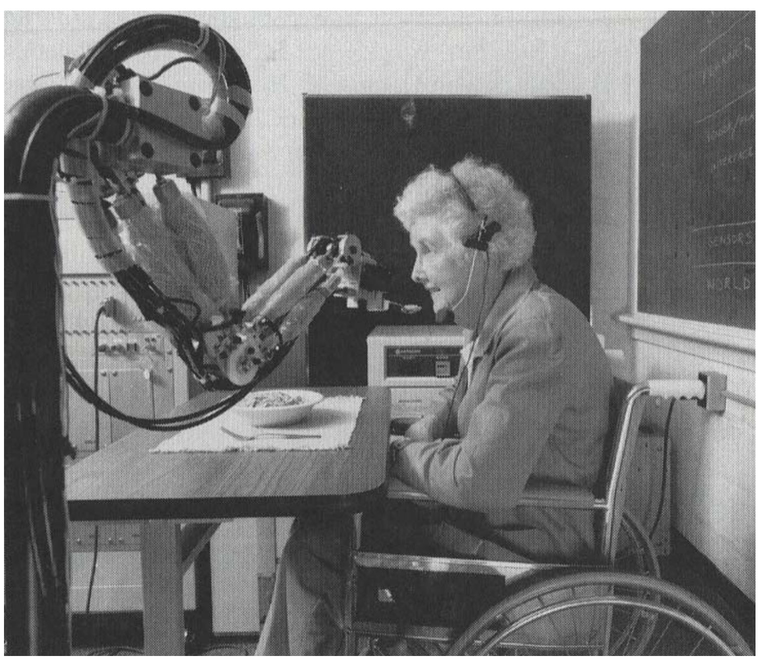
$-l_\varepsilon/l_p$ steady-state gain



Robotic Rubbertuator and Load[5] (McKibben Artificial Muscle)

$$G(s)=\frac{X(s)}{P(s)}=\frac{10}{s^2+10s+29}$$

$p(t)$ input air pressure $x(t)$ output displacement

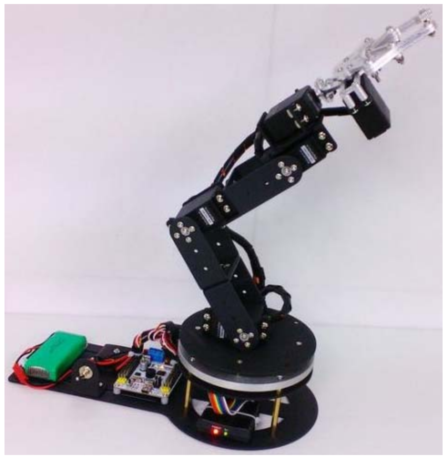


Robotic Swivel[5]

$$G(s) = \frac{\Omega(s)}{V(s)} = \frac{K}{(s^2 + 4s + 10)}$$

$v(t)$ input voltage

$\omega(t)$ output swivel velocity



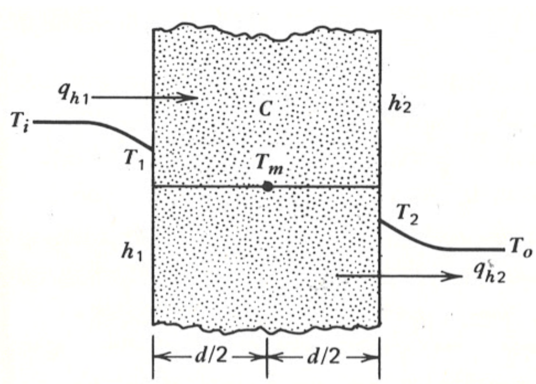
Heat Transfer System[5]

$$G(s) = \frac{T_o(s)}{T_i(s)} = \frac{1}{RCs + 1}$$

$t_i(t)$ input temperature

$t_o(t)$ output temperature

RC thermal time constant



Pneumatic System[5]

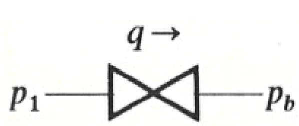
$$G(s) = \frac{P_b(s)}{P_1(s)} = \frac{1}{RCs + 1}$$

$p_1(t)$ input pressure

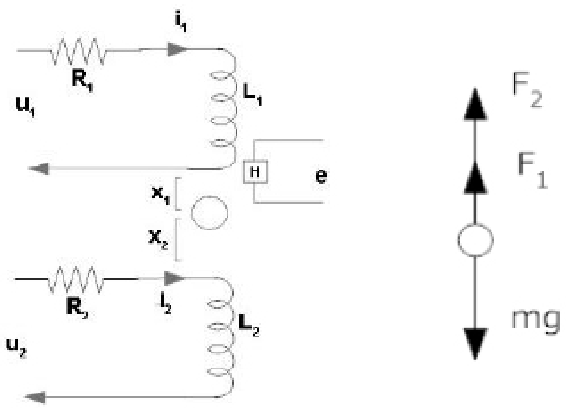
$p_b(t)$ output pressure

q mass flow rate

RC pneumatic time constant



Magnetically-Levitated Ball[6]



$$G(s) = \frac{X(s)}{I(s)} = \frac{-1}{As^2 - B}$$

$x(t)$ output ball height
$i(t)$ input current

$$A = \frac{i_0}{2g} \qquad B = \frac{i_0}{x_0}$$

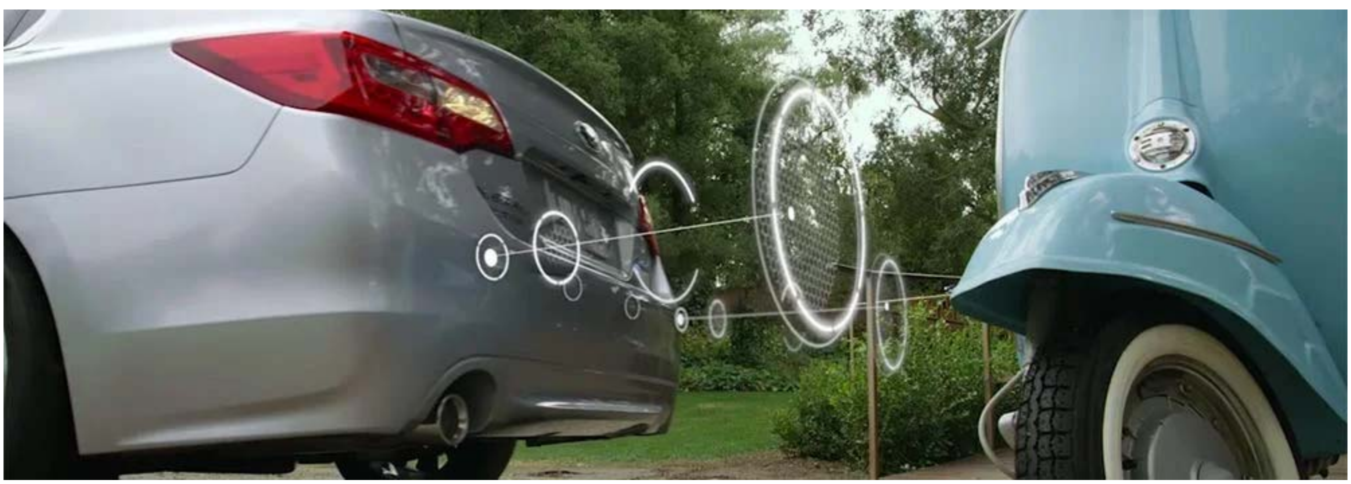
i_0 and x_0 are the nominal current and ball height at which the linearization was performed

Autonomous Automobile Backup[Dr. Bob]

$$G(s) = \frac{X(s)}{F(s)} = \frac{K}{s(ms+c)}$$

$f(t)$ road force input $\qquad$ $x(t)$ backup distance

K open-loop gain
m car mass
c effective viscous damping coefficient



[What is Subaru Reverse Automatic Braking? (goldsteinsubaru.com)]

[6] H. Huang, H. Du & W. Li, 2015, "Stability enhancement of magnetic levitation ball system with two controlled electromagnets," Power Engineering Conference (AUPEC), 2015 Australasian Universities: 1-6.

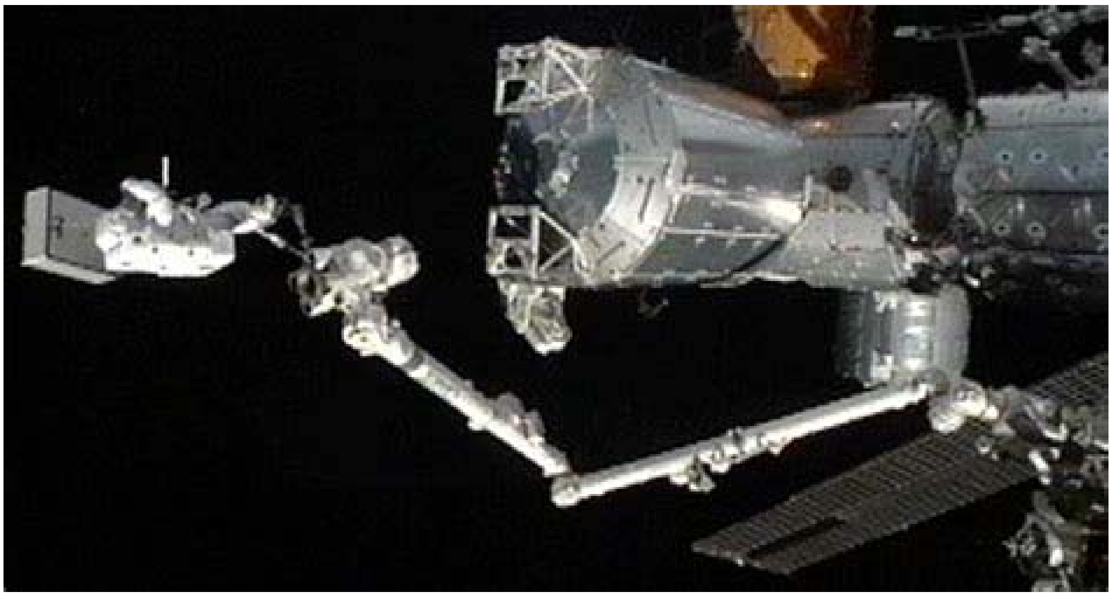
Third-Order and Fourth-Order Systems

suitable for ME 3012 Term Projects using an internal pre-filter $G_{Pi}(s)$

Flexible Robot Control[3]

$$G(s) = \frac{\Theta(s)}{\mathrm{T}(s)} = \frac{s+500}{s(s+0.03)(s^2+2.57s+6667)}$$

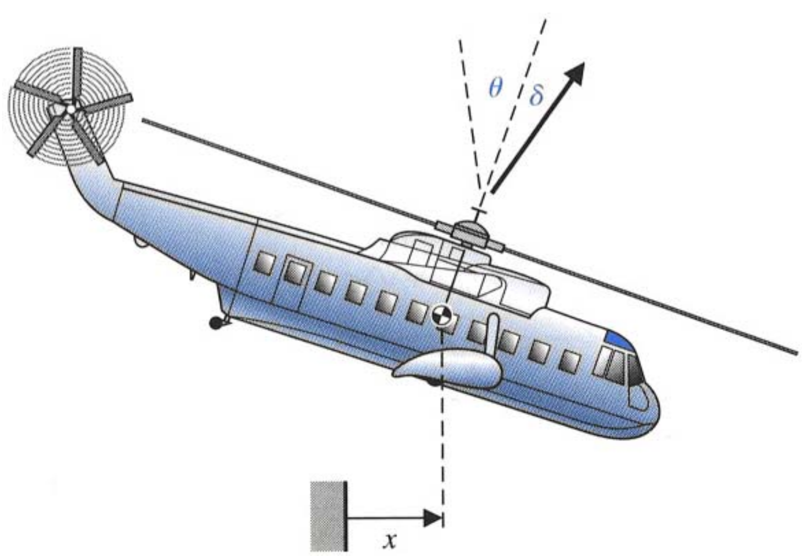
$\tau(t)$ torque input $\theta(t)$ angle output



Helicopter Pitch Control[3]

$$G(s) = \frac{\Theta(s)}{\mathrm{T}(s)} = \frac{25(s+0.03)}{(s+0.4)(s^2-0.36s+0.16)}$$

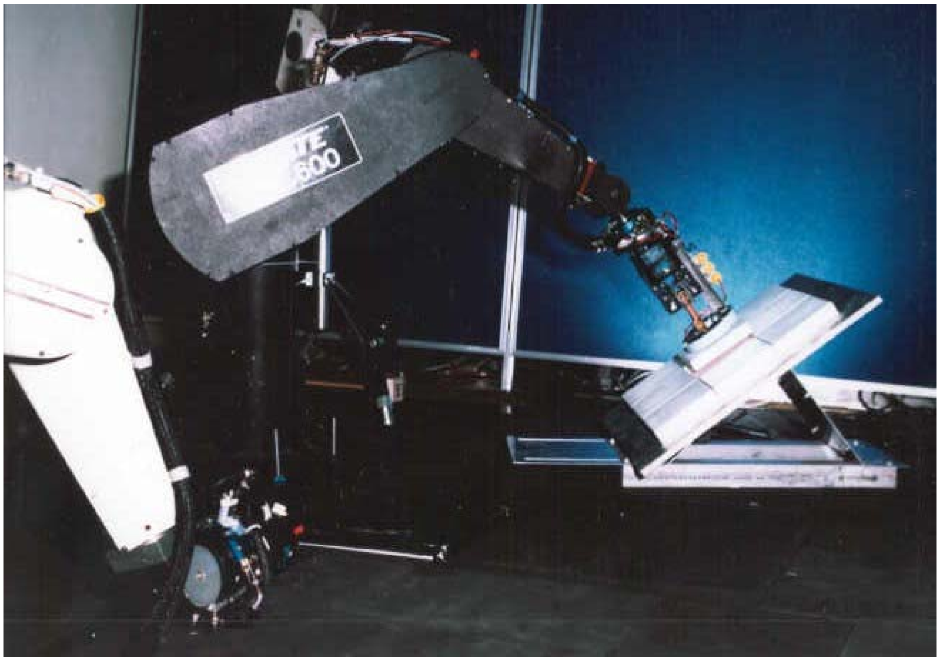
$\tau(t)$ torque input $\theta(t)$ output pitch angle



Robot Force Control[3]

$$G(s) = \frac{F(s)}{\mathrm{T}(s)} = \frac{K(s+2.5)}{(s^2+2s+2)(s^2+4s+5)}$$

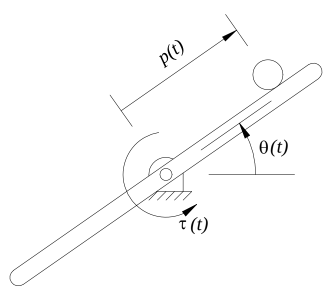
$\tau(t)$ torque input $f(t)$ force output



Third-Order and Fourth-Order Systems

less suitable for ME 3012 Term Projects

Ball-and-Beam System[1]



$$\left[\frac{J_b}{r^2}+m\right]\ddot{p}(t)+m\,g\sin\theta(t)-m\,p(t)\,\dot{\theta}(t)^2 = 0$$

$$\left[m\,p(t)^2+J+J_b\right]\ddot{\theta}(t)+2\,m\,p(t)\,\dot{p}(t)\,\dot{\theta}(t)+m\,g\;p(t)\cos\theta(t) = \tau(t)$$

non-linear

$$\left[\frac{J_b}{r^2}+m\right]\ddot{p}(t)+m\,g\theta(t) = 0$$

$$\left[\frac{mL^2}{4}+J+J_b\right]\ddot{\theta}(t)\;+m\,g\;p(t) = \tau(t)$$

linearized

L is the constant half-length of the beam, m and r are the mass and radius of the ball, respectively, J_b and J are the mass moment of inertia of the ball and beam, respectively.

In order to derive the overall SISO transfer function for the ball-and-beam system, take the Laplace Transform of both sides of both of the linearized ODEs above. Then use algebra to eliminate $\Theta(s)$ between the two equations and arrive at $G_1(s)$. This process yields the following Type 0, fourth-order, unstable open-loop transfer function:

$$G_1(s)=\frac{P(s)}{T(s)}=\frac{-mg}{J_{E1}J_{E2}s^4-m^2g^2}$$

where: $J_{E1}=\left[\frac{J_b}{r^2}+m\right]$ $\quad J_{E2}=\left[\frac{mL^2}{4}+J+J_b\right]$

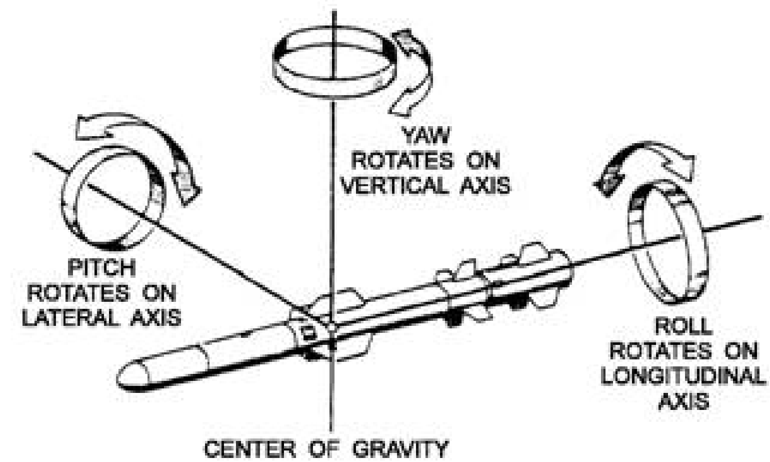
This process could alternatively eliminate $P(s)$ between the two equations and arrive at $G_2(s)$, the following Type 0, fourth-order, unstable open-loop transfer function:

$$G_2(s)=\frac{\Theta(s)}{T(s)}=\frac{J_{E1}s^2}{J_{E1}J_{E2}s^4-m^2g^2}$$

Missile Yaw Control[3] (cannot use internal pre-filter for positive poles)

$$G(s) = \frac{\mathrm{A}(s)}{\mathrm{T}(s)} = \frac{-0.5(s^2 - 2500)}{(s-3)(s^2 + 50s + 1000)}$$

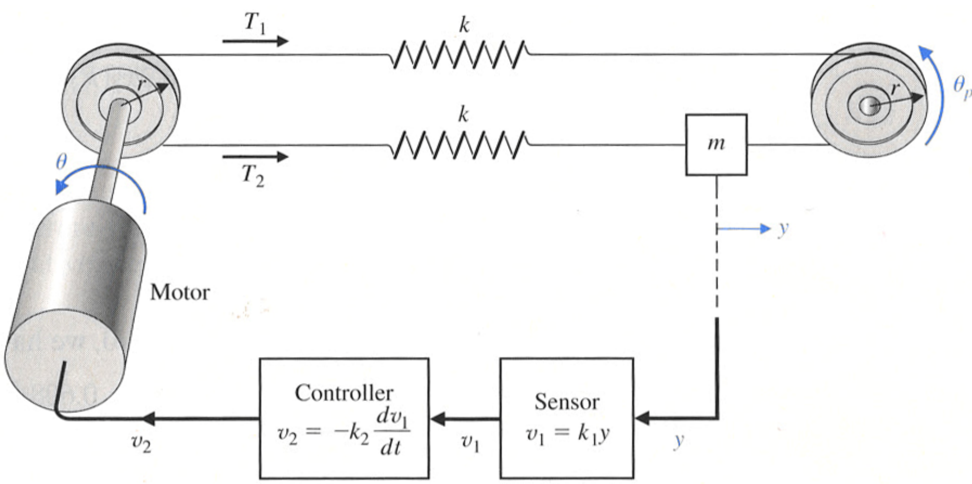
$\tau(t)$ torque input $\alpha(t)$ yaw acceleration

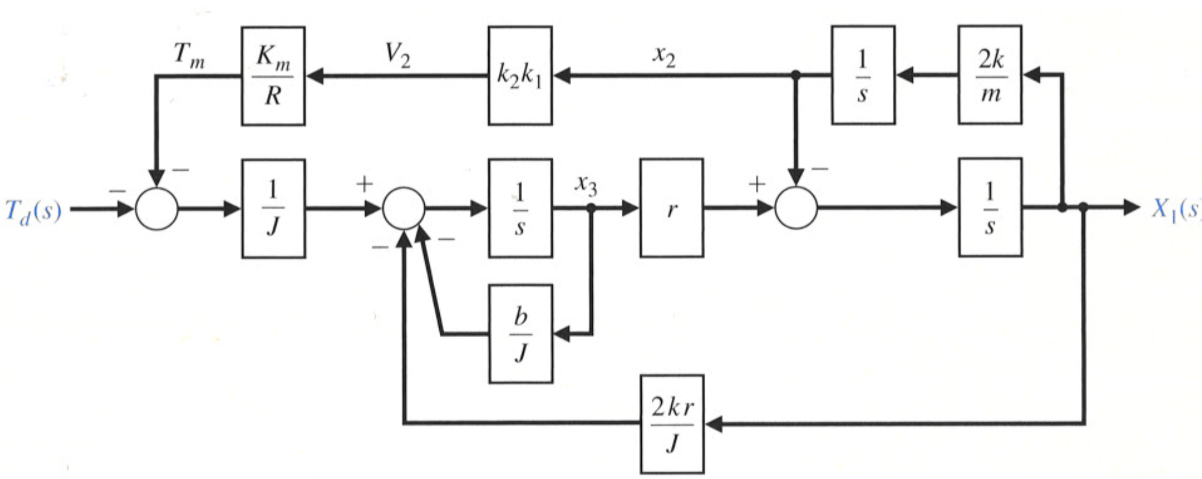


Printer Belt Drive[3] (closed-loop)

$$T(s)=\frac{X_1(s)}{T_d(s)}=\frac{-\left(\frac{r}{J}\right)s}{s^3+\left(\frac{b}{J}\right)s^2+2k\left(\frac{1}{m}+\frac{r^2}{J}\right)s+\frac{2k}{mJ}\left(b+\frac{K_m k_1 k_2 r}{R}\right)}$$

$\tau_d(t)$ disturbance torque input $\qquad x_1(t)=r\theta(t)-y(t)$ output displacement error





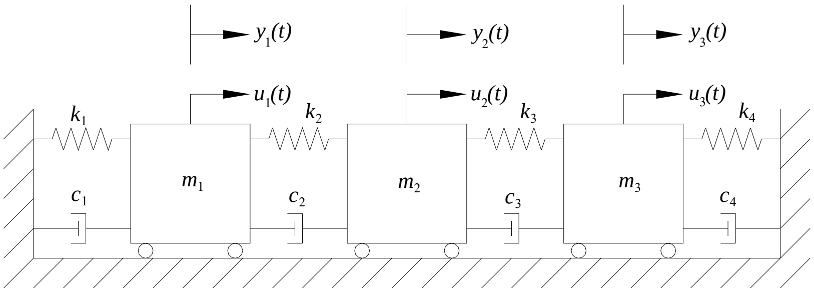
m	mass	0.2 kg
k_1	light sensor	1 V/m
k_2	velocity feedback gain	0.1 Vs/m
r	radius	0.15 m
L	inductance	0 (ignore)
b	rotational viscous damping	0.25 Nms/rad
R	resistance	2 Ω
K_m	torque constant	2 Nm/A
$J = J_{\text{motor}} + J_{\text{pulley}}$	inertia	0.01 kg-m^2

6. Advanced Real-World Models

less suitable for ME 3012 Term Projects

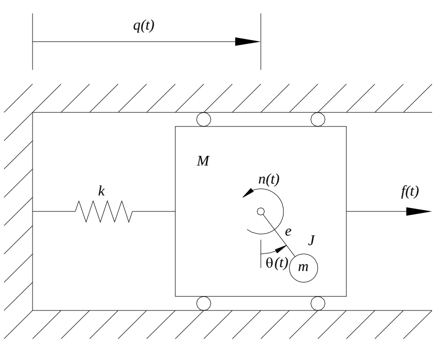
(use state-space controller design techniques instead of classical methods)

Three-dof translational mechanical system[1]



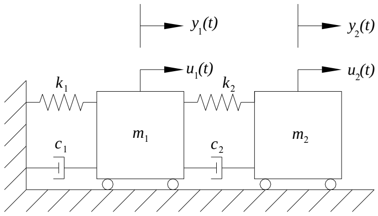
$$
\begin{aligned}
m_1\ddot{y}_1(t)+(c_1+c_2)\dot{y}_1(t)+(k_1+k_2)y_1(t)-c_2\dot{y}_2(t)-k_2y_2(t) &= u_1(t)\\
m_2\ddot{y}_2(t)+(c_2+c_3)\dot{y}_2(t)+(k_2+k_3)y_2(t)-c_2\dot{y}_1(t)-k_2y_1(t)-c_3\dot{y}_3(t)-k_3y_3(t) &= u_2(t)\\
m_3\ddot{y}_3(t)+(c_3+c_4)\dot{y}_3(t)+(k_3+k_4)y_3(t)-c_3\dot{y}_2(t)-k_3y_2(t) &= u_3(t)
\end{aligned}
$$

Non-linear Proof-Mass Actuator System[1]



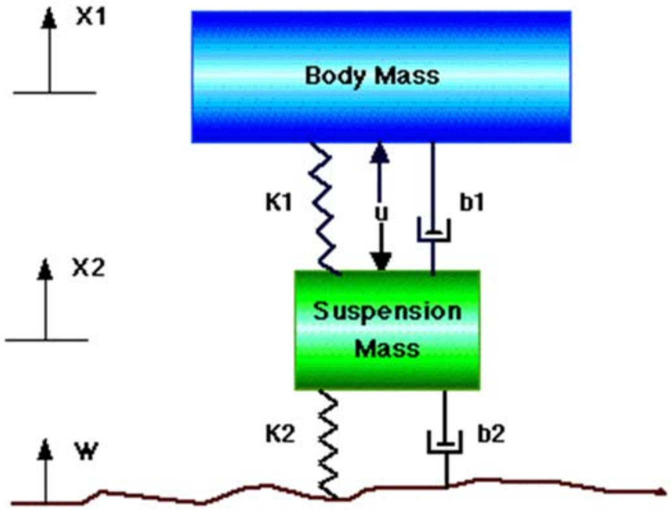
$$
\begin{aligned}
(M+m)\ddot{q}(t)+kq(t)+me(\ddot{\theta}(t)\cos\theta(t)-\dot{\theta}^2(t)\sin\theta(t)) &= 0\\
(J+me^2)\ddot{\theta}(t)+me\ddot{q}(t)\cos\theta(t) &= n(t)
\end{aligned}
$$

Two-dof translational mechanical system[1]



$$
\begin{aligned}
m_1\ddot{y}_1(t)+(c_1+c_2)\dot{y}_1(t)+(k_1+k_2)y_1(t)-c_2\dot{y}_2(t)-k_2y_2(t) &= u_1(t)\\
m_2\ddot{y}_2(t)+c_2\dot{y}_2(t)+k_2y_2(t)-c_2\dot{y}_1(t)-k_2y_1(t) &= u_2(t)
\end{aligned}
$$

Automobile Suspension System[2]



$$m_1\ddot{x}_1(t)+b_1(\dot{x}_1(t)-\dot{x}_2(t))+k_1(x_1(t)-x_2(t)) = u(t)$$
$$m_2\ddot{x}_2(t)+b_2(\dot{x}_2(t)-\dot{w}(t))+k_2(x_2(t)-w(t))+c_2\dot{y}_2(t)-b_1(\dot{x}_1(t)-\dot{x}_2(t))-k_1(x_1(t)-x_2(t)) = -u(t)$$

$$G_u(s)=\frac{X_1(s)-X_2(s)}{U(s)}=\frac{(m_1+m_2)s^2+b_2s+K_2}{(m_1s^2+b_1s+K_1)(m_2s^2+(b_1+b_2)s+(K_1+K_2))-(b_1s+K_1)^2}$$

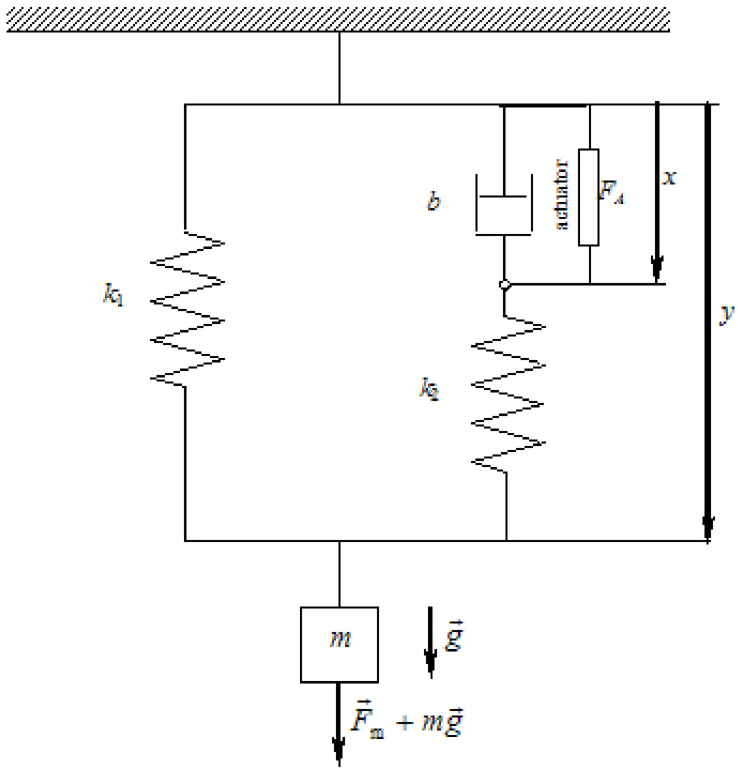
$$G_w(s)=\frac{X_1(s)-X_2(s)}{W(s)}=\frac{-m_1(b_2s+K_2)s^2}{(m_1s^2+b_1s+K_1)(m_2s^2+(b_1+b_2)s+(K_1+K_2))-(b_1s+K_1)^2}$$

In this model, the Body Mass is m_1, generally one-fourth of the vehicle mass (excluding tires). m_2 is the Suspension Mass, the mass of one tire. As seen in the above transfer functions, this model is 4th-order.

To make this model fit the second-order system controller designs we focus on in ME 3012:

- Eliminate the tire mass (m_2, the Suspension Mass) and include half of this into m_1.
- Then combine the two springs and two dampers in series to obtain one equivalent spring stiffness and one equivalent damper coefficient.

Human Skeletal Muscle Model[7]



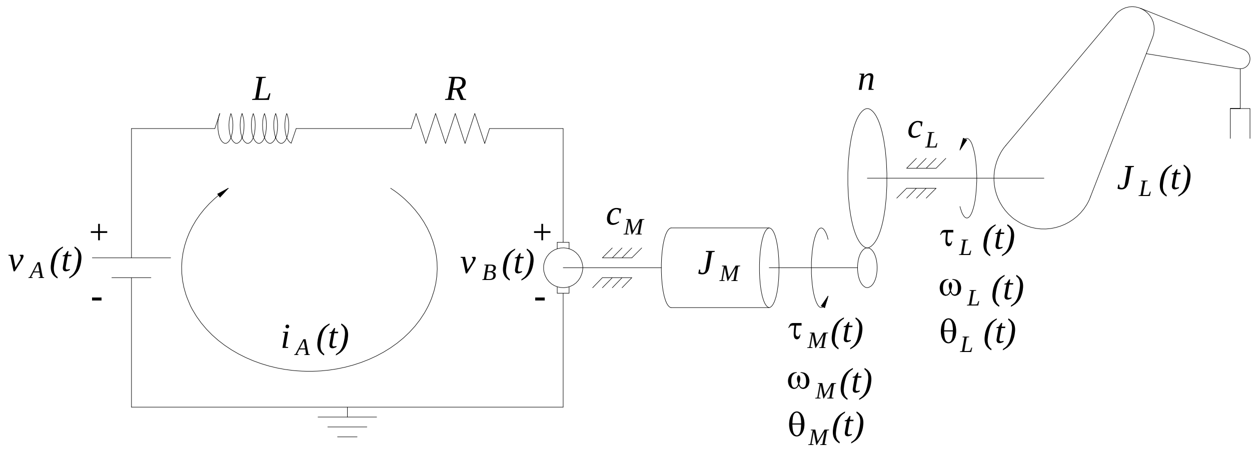
$$\dddot{y}(t)+\frac{k_2}{b}\ddot{y}(t)+\frac{(k_1+k_2)}{m}\dot{y}(t)+\frac{k_1k_2}{mb}y(t) = \frac{\dot{F}_m(t)}{m}+\frac{k_2}{mb}(F_A(t)+F_m(t)+mg+k_1L_{1R})$$

where:

$y(t)$	absolute displacement of the muscle end
m	lumped muscle mass
k_1	linear spring stiffness representing the muscle fascia, parallel elastic component
k_2	linear spring stiffness representing the connecting tendons, series elastic component
b	viscous damping coefficient representing the muscle energy loss
$F_A(t)$	muscle actuation force (the integrated effects of all contracting sarcomeres)
$F_m(t)$	external load applied to the muscle
g	acceleration due to gravity
L_{1R}	neutral length of the muscle (the unstretched length of spring k_1).

[7] Dr. Bob's ME 4670 / 5670 Biomechanics NotesBook Supplement, derived by Elvedin Kljuno

Armature circuit / DC servomotor / gear box / robot joint[1] – ME 3012 Term Example



$$LJ\dddot{\theta}_{\mathrm{L}}(t)+(Lc+RJ)\ddot{\theta}_{\mathrm{L}}(t)+(Rc+K_{\mathrm{T}}K_{\mathrm{B}})\dot{\theta}_{\mathrm{L}}(t) = \frac{K_{\mathrm{T}}}{n}v_{\mathrm{A}}(t)$$

$$LJ\ddot{\omega}_{\mathrm{L}}(t)+(Lc+RJ)\dot{\omega}_{\mathrm{L}}(t)+(Rc+K_{\mathrm{T}}K_{\mathrm{B}})\omega_{\mathrm{L}}(t) = \frac{K_{\mathrm{T}}}{n}v_{\mathrm{A}}(t)$$

where $J = J_{\mathrm{M}} + \frac{J_{\mathrm{L}}}{n^2}$ and $c = c_{\mathrm{M}} + \frac{c_{\mathrm{L}}}{n^2}$ are the effective polar inertia and viscous damping coefficient reflected to the motor shaft.

Numerical Parameters

Parameter	Value	Units	Name
L	0.0006	H	armature inductance
R	1.40	Ω	armature resistance
k_B	0.00867	V/deg/s	motor back emf constant
J_M	0.00844	lb_f-in-s^2	motor shaft polar inertia
b_M	0.00013	lb_f-in/deg/s	motor shaft damping constant
k_T	4.375	lb_f-in/A	torque constant
n	200	unitless	gear ratio
J_L	1	lb_f-in-s^2	load shaft polar inertia
b_L	0.5	lb_f-in/deg/s	load shaft damping constant

$$G_{\omega}(s) = \frac{\Omega_{\mathrm{L}}(s)}{V_{\mathrm{A}}(s)} = \frac{K_{\mathrm{T}}/n}{LJs^2+(Lc+RJ)s+(Rc+K_{\mathrm{T}}K_{\mathrm{B}})} = \frac{5}{s^2+11s+1010}$$

$$G_{\theta}(s) = \frac{\Theta_{\mathrm{L}}(s)}{V_{\mathrm{A}}(s)} = \frac{K_{\mathrm{T}}/n}{s(LJs^2+(Lc+RJ)s+(Rc+K_{\mathrm{T}}K_{\mathrm{B}}))} = \frac{5}{s(s^2+11s+1010)}$$